Recall

$X, Y \neq \phi$

$\text{Card}(X) \geq \text{Card}(Y)$ $\exists f : X \to Y$ surjection

$\text{Card}(X) \leq \text{Card}(Y)$ $\exists f : X \to Y$ injection

$\text{Card}(X) = \text{Card}(Y)$ $\exists f : X \to Y$ bijection

R1 $\text{Card}(X) \leq \text{Card}(Y)$ iff $\text{Card}(Y) \geq \text{Card}(X)$

R2 $X, Y \neq 0$ then $\text{Card}(X) \leq \text{Card}(Y)$
or $\text{Card}(Y) \leq \text{Card}(X)$

Proof:

Schröder-Bernstein If $\text{Card}(X) \le \text{Card}(Y)$ & $\text{Card}(Y) \le \text{Card}(X)$

then $\text{Card}(X) = \text{Card}(Y)$.

Proof:- Done in A classes. $\Box$

$$I = \{ f : A \to B \mid f \text{ is an injection}, A \subseteq X \text{ and } B \subseteq Y \}$$

$\boxed{I \neq \phi}$ $A = \{x_0\}$ and $B = \{y_0\}$ for some $x_0 \in X$ and $y_0 \in Y$ $f : A \to B$ $f(x_0) = y_0$

I is subsets of $X \times Y$. I is a partially ordered set "$\subseteq$"

It is easy to see, by Zorn's lemma, I has a maximal element

f - maximal element $\quad \mathrm{Dom}(f) = A \quad , \mathrm{Range}(f) = B$.

Suppose $A \neq X$ & $B \neq Y$

$x_0 \in X \setminus A \quad$ and $\quad y_0 \in Y \setminus B \quad f(x_0) := y_0$

$\Rightarrow \quad f$ was not maximal

Hence either $A = X \Rightarrow \mathrm{Card}(X) \leq \mathrm{Card}(Y)$

$B = Y \Rightarrow$ look at $f^{-1} \Rightarrow \mathrm{Card}(Y) \leq \mathrm{Card}(X)$

which is injection

$Y \to X \quad \square$

$X \neq \emptyset \quad \text{Card}(X) < \text{Card}(\mathcal{P}(X))$

$\emptyset$: known to greeks \ {Sina} $\square$.

$$[n] = \{ i \mid 1 \leq i \leq n, i \in \mathbb{N} \}$$

$\text{Card}(X) = \text{Card}([n])$

Then we will say X is finite and has n elements

X is countable if $\text{Card}(X) \leq \text{Card}(\mathbb{N})$

X & Y are countable $\quad X \times Y$ is countable.

$f: X \to \mathbb{N}$ injection $\quad h: X \times Y \to \mathbb{N} \times \mathbb{N}$

$g: Y \to \mathbb{N}$ injection $\quad h(x,y) = (f(x), g(y))$ an injection

$\Rightarrow \text{card}(X \times Y) \leq \text{card}(\mathbb{N} \times \mathbb{N})$

$f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$(m,n) \to 3^m 2^n$ $\quad \Rightarrow \text{card}(X \times Y) \leq \text{card}(\mathbb{N})$ $\quad$ D

(injection

$\Rightarrow \text{card}(\mathbb{N} \times \mathbb{N}) \leq \text{card}(\mathbb{N})$.

$f: \mathcal{P}(\mathbb{N}) \to \mathbb{R}$

$$f(A) = \begin{cases} \sum\limits_{n \in A} 2^{-n} & \mathbb{N} \setminus A \text{ infinite} \\ 1 + \sum\limits_{n \in A} 2^{-n} & \mathbb{N} \setminus A \text{ finite} \end{cases}$$

✓

$\boxed{\text{injection}}$

Define: $\text{Card}(X) = \mathcal{C}$ iff $\text{Card}(X) = \text{Card}(\mathbb{R})$

R 6. $\text{Card}(\mathcal{P}(\mathbb{N})) = \mathcal{C}$

$g : \mathcal{P}(\mathbb{Z}) \to \mathbb{R}$

$$g(A) = \begin{cases} \log\left(\sum\limits_{n \in A} 2^{-n}\right) & A \text{ is bounded below} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Card}(\mathbb{R}) = \mathrm{Card}((0,1))$$

$$f(x) = 1 - \frac{1}{x^2}$$

$$f(x) = \frac{1}{x^2}$$

$$f(x) = \left(\frac{3/4 - \frac{1}{4}}{2+2}\right)(x-2) + 3/4$$

$$= \frac{1}{8}x + \frac{1}{2}$$