$f : \mathbb{R} \to \mathbb{R}$

3

(*) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x)| < \varepsilon$ whenever $|x| < \delta$

local property

implication

$\Leftarrow$

(a) For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$ $|x| < \delta \Rightarrow |f(x)| < \varepsilon$

qualified better

(whenever)

(a) $\Leftrightarrow$ (*)

(b)

[global p

False $b \Rightarrow$ (*)

(*) $\Rightarrow$

$f : \mathbb{R} \to$

$f(x)$

(b) For every $\delta > 0$ $\exists \varepsilon > 0$ such that for all $x \in \mathbb{R}$ $|x| < \delta$ implies $|f(x)| < \varepsilon$

[global property]

$\boxed{\text{False}}$ $b \Rightarrow \neg (*)$

$(*) \Rightarrow \neg b$

OPINION POLL

T	F	F	T
T	F	T	F
0	13	16	1

$\varepsilon = 2$ $|f(x)| \geq \varepsilon$ $\forall x \in \mathbb{R}$

$\Rightarrow$ there is no $\delta > 0$ such that $|f(x)| < \varepsilon$ whenever $|x| < \delta$.

$(*)$ does not hold.

T

$x \in \mathbb{R}$ & $|x| < \delta$

$\Downarrow$

$\forall \delta > 0$ $|f(x)| < 10$

$|f(x)| < 10$

$\varepsilon = 10$ and (b) holds

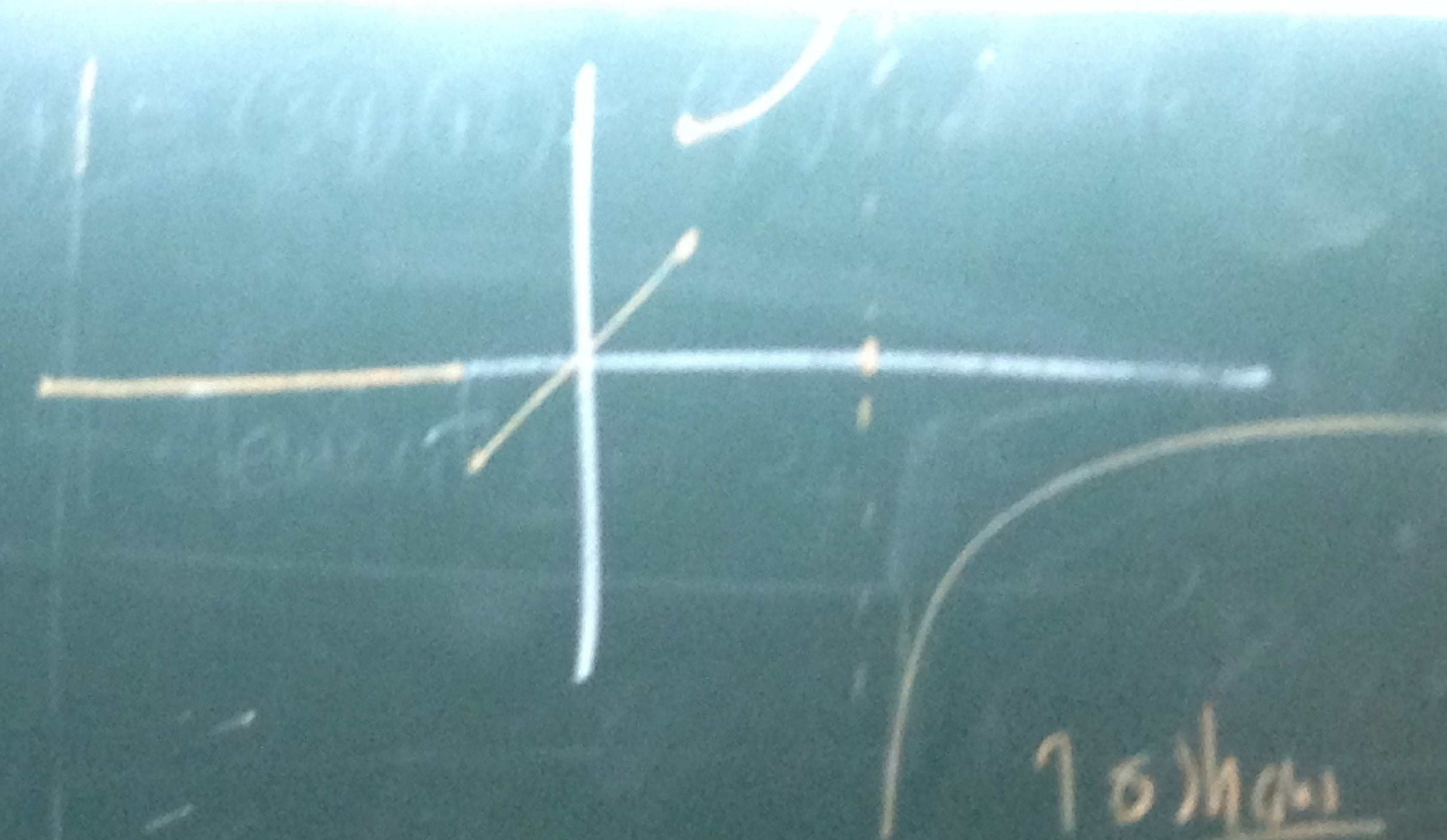
$f : \mathbb{R} \to \mathbb{R}$

$f(x) = 3$

ε 10

3

$$f: \mathbb{R} \to \mathbb{R} \qquad f(x) = \begin{cases} 0 & x < -\frac{1}{2} \\ x & -\frac{1}{2} \le x \le \frac{1}{2} \\ \frac{1}{1-x} & -\frac{1}{2} < x \ \& \ x \neq 1 \\ 0 & x = 1 \end{cases}$$



(*) to hold

let $\varepsilon > 0$ be given.

take $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$

$|x| < \delta \Rightarrow |x| < \frac{1}{2}$

$\Rightarrow f(x) = x$

$\Rightarrow |f(x)| < \delta < \varepsilon$

b not hold $\left[\exists \delta > 0 \ \forall \varepsilon > 0 \ \exists x \in \mathbb{R} \ |x| < \delta \ \& \ |f(x)| > \varepsilon \right.$

To show

$\delta = 2$.

let $\varepsilon > 0$ be given

$x = 1 - \frac{1}{2\varepsilon} \Rightarrow$ (Ex) $|f(x)| > \varepsilon$.

$\frac{1}{2} < x < 2$

As $\varepsilon > 0$ was arbitrary (*) holds.

Q $f : \mathbb{R} \to \mathbb{R}$

f - differentiable.

$f'(0) > 0$

$\exists\, \delta > 0$ in $(-\delta, \delta)$, f is increasing ?

T

F - 5

X

C

Example

(*) For every $\varepsilon > 0$ There exists $\delta > 0$ s.t. $|f(x)| < \varepsilon$ whenever $|x| < \delta$

$\delta = \delta(\varepsilon)$

(C) $\exists\, \delta > 0$ st $\forall\, \varepsilon > 0$ & $\forall x \in \mathbb{R}$, $|x| < \delta \Rightarrow |f(x)| < \varepsilon$

Example: $f(x) = x$ show (*) holds

let $\varepsilon > 0$ be given

Take $\delta = \varepsilon$

$$|x| < \delta \overset{\delta = \varepsilon}{\Longrightarrow} |x| < \varepsilon \overset{\text{As } f(x) = x}{\Longrightarrow} |f(x)| < \varepsilon.$$

As $\varepsilon > 0$ was arbitrary

$\Rightarrow$ (*) holds $\square$

show (c) does not hold for f

$\neg$(c) $\forall\, \delta > 0 \quad \exists\, \varepsilon > 0$ & $\exists\, x \in \mathbb{R}$ $|x| < \delta$ & $|f(x)| \geq \varepsilon$.

let $\delta > 0$ be given.

Take $\varepsilon = \dfrac{\delta}{2}$

$\mathbb{R}$-order

$f(x) = x$

Take $x = \dfrac{3\delta}{4} \implies |x| < \delta \implies |f(x)| > \delta/2$

$|x| > \delta/2$

$\varepsilon = \delta/2$

$\implies |f(x)| > \varepsilon$

$\implies |f(x)| \geq \varepsilon$.

$> \varepsilon \implies >$ or $= \varepsilon$

(*)' $\forall \varepsilon > 0 \ \exists \delta > 0$ st $|f(x)| < \varepsilon$, $0 \neq |x| < \delta$

(d) For every $\varepsilon > 0$ & for all $x \in \mathbb{R}$
$\exists \delta > 0$ st $|x| < \delta \Rightarrow |f(x)| < \varepsilon$

(*) $\forall \varepsilon > 0 \ \exists \delta > 0$ st $|f(x)| < \varepsilon$
whenever $|x| < \delta$

If (*) holds take $x = 0$
$\Rightarrow |f(0)| < \varepsilon \quad \forall \varepsilon > 0$
$\Rightarrow f(0) = 0 \Rightarrow$ (d) holds $\square$

(d) holds for $f:\mathbb{R}\to\mathbb{R}$

Claim: $f(0)=0$

let $\varepsilon>0$ be given

$\exists\ \delta>0 \qquad |x|<\delta \Rightarrow |f(x)|<\varepsilon$

ake $x=0 \qquad |0|<\delta \Rightarrow |f(0)|<\varepsilon$

ε- arbitrary $\Rightarrow |f(0)|<\varepsilon \quad \forall\ \varepsilon>0$

$f(0)=0$.

Claim 2: $f:\mathbb{R}\to\mathbb{R}$

$f(0)=0$

then (d) ho

Pf: $\boxed{\text{let } \varepsilon>0}$

. $x=0$, $f(0)=$

$\underline{(\forall\delta>0\ \ 0<\delta)}$ $|f(0)|<\varepsilon$

$\boxed{\text{let } x\neq 0 \text{ be given}}$

Take $\delta=\frac{|x|}{2}$

$\delta > 0$ & $|x| < \delta \Rightarrow$ False ($\because$ $|x| < \frac{|x|}{2}$

$|x| < \delta \Rightarrow |f(x)| < \varepsilon$. ∎

Take $f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$ (d) holds (*) does not hold.

Take $f(x) = \begin{cases} x & x \neq 0 \\ 0 & x = 0 \end{cases}$ (*)' holds (d) does not hold.