**Worksheet-Part 2**

1. **Due Date: March 17th, 2023, 12pm**
Let $(\Gamma, \mu)$ be a weighted graph that is connected and locally finite. Let $S_n$ be a simple symmetric random walk on $\mathbb{Z}$. Assume the following:
**Result:** *Let for* $-1 \leq t \leq 1$, $k \in \mathbb{N} \cup \{0\}$

$$H_k(t) = \frac{1}{2}(t + i(1-t^2)^{\frac{1}{2}})^k + \frac{1}{2}(t - i(1-t^2)^{\frac{1}{2}})^k$$

*be the Chebyshev polynomial. Then*

$$P^n = \sum_{k\in\mathbb{Z}} \mathbb{P}(S_n = k) H_{|k|}(P) \qquad \text{and } \| H_{|k|}(P) f \|_2 \leq \| f \|_2, \tag{3}$$

*for any* $f \in C(V)$.

   (a) Let $x, y \in V$ and $f(z) = \frac{1_x(z)}{\mu_x}$, $g(z) = \frac{1_y(z)}{\mu_y}$. Show that

$$p_k(x,y) = \frac{1}{(\mu_x \mu_y)^{\frac{1}{2}}} \langle f, P^k g \rangle$$

   (b) Using the Result show that

$$\langle f, P^k g \rangle = \sum_{k\in\mathbb{Z}} \mathbb{P}(S_n = k) \langle f, H_{|k|}(P) g \rangle \leq \sum_{|k| \geq d(x,y)} \mathbb{P}(S_n = k) \| f \|_2$$

   (c) Show that

$$\mathbb{P}(|S_n| \geq d(x,y)) \leq 2 \exp(-\frac{d(x,y)^2}{n})$$

   (d) Conclude that

$$p_n(x,y) \leq \frac{2}{(\mu_x \mu_y)^{\frac{1}{2}}} \exp(-\frac{d(x,y)^2}{n}).$$

*Hint:* This is [Bar17, Theorem 4.9]. Please write out details in sequential manner as required. Do not just replicate the order of proof in [Bar17].

# References

[Bar17] Martin T Barlow. *Random walks and heat kernels on graphs*, volume 438. Cambridge University Press, 2017.