Worksheet-Part 1

1. **Due Date: March 17th, 2023, 1030am**
   Let (Γ, μ) be a weighted graph that is connected and locally finite. Let X_n be a random walk on (Γ, μ) and $p_n(\cdot, \cdot)$ be the n-th step transition density for $n \geq 1$ with P being the one step transition operator.

   (a) Fix $x \in V$ and $n \geq 1$. Let $r_n^x(\cdot) = p_n(x, \cdot) + p_{n+1}(x, \cdot)$ and $\phi_n(x) = r_{2n}^x(x)$. Assume Γ satisfies (N_α) for $\alpha \geq 1$

      i. Show that
      $$\phi_{n+1}(x) - \phi_n(x) = -\mathcal{E}(r_n^x, r_n^x) \leq -2^{-\frac{4}{\alpha}} C_N \phi_n(x)^{1+\frac{2}{\alpha}} \tag{1}$$

      ii. Conclude that there exists $c_1 > 0$ such that
      $$\phi_{n+1}(x) - \phi_n(x) \leq c_1 \phi_n(x)^{1+\frac{2}{\alpha}}$$
      and that this implies that there exists $c_2 > 0$ such that
      $$\phi_n(x) \leq c_2^{-\frac{\alpha}{2}} n^{-\frac{\alpha}{2}} \tag{2}$$

   (b) Show that if Γ satisfies (N_α) for $\alpha \geq 1$ then there exists $c_3 > 0$ such that
   $$p_n(x, x) \leq \frac{c_3}{\max(1, n)^{\frac{\alpha}{2}}}$$
   for all $x \in V$ and $n \geq 0$.

   Hint: This is [Bar17, Theorem 4.3 (a) $\implies$ (b)]. Please write out details in sequential manner as required. Do not just replicate the order of proof in [Bar17].