Homework Set 7
Due Date: April 8th, 2023, 8am.
Submission on Moodle.

1. Let $\{V_i : i \geq 1\}$ be independent exponential random variables with mean λ,

$$T_k = \sum_{i=1}^{k} V_i \qquad N_t = \sum_{k=1}^{\infty} 1(T_k \leq t).$$

   (a) Show that for each $t \geq 0$, $\mathbb{P}(N_t < \infty) = 1$.

   (b) N_t has a Poisson distribution with parameter λt.

   (c) Let $0 = t_0 \leq t_1 \leq \ldots \leq t_m$. Show that the collection of random variables $\{N_{t_i} - N_{ti-1} : 1 \leq i \leq n\}$ are mutually independent and $N_{t_i} - N_{ti-1} \sim$ Poisson $(\lambda(t_i - t_{i-1}))$

2. We place M marbles in P pots. At each time unit we choose one of the marbles uniformly at random and place it in one of the urns also uniformly chosen at random. Denote by M_n to be the number of marbles in the first urn at time n. Find a_n, b_n, so that $a_n M_n + b_n$ is a martingale.

3. Let $V = \mathbb{Z}^2$ and $E = \{\{\mathbf{i}, \mathbf{j}\} : \mathbf{i}, \mathbf{j} \in V \text{ and } \| \mathbf{i} - \mathbf{j} \| \leq \sqrt{2}\}$ with $\| \cdot \|$ denoting the euclidean norm. Consider the standard weight function $\mu \equiv 1$ on E. Let $\{X_n\}_{n\geq 1}$ be the random walk on this weighted graph and $\mathcal{A}_n$ be the events observable by time n w.r.t $\{X_k\}_{k\geq 1}$.

   (a) Show that $M_n = \| X_n \|^2 - \frac{3}{2}n$ is a martingale w.r.t the filtration $\mathcal{A}_n$.

   (b) Let $\tau_R = \min\{n \geq 0 : \| X_n \|^2 \geq R^2\}$. Show that

$$\frac{2}{3}R^2 \leq \mathbb{E}[\tau_R] \leq \frac{2}{3}(R + \sqrt{2})^2$$

4. Let X_n be a Markov chain on S with transition matrix P and initial distribution μ. Assume S is finite and $\mathcal{A}_n$ are observable events by time n w.r.t $\{X_k\}_{k\geq 1}$. Find conditions on a and $f : S \to \mathbb{R}$ such that $M_n = a^{-n} f(X_n)$ is a martingale w.r.t the filtration $\mathcal{A}_n$.

5. Consider a martingale where Z_n can take on only the values 2^{-n-1} and $1 - 2^{-n-1}$, each with probability $\frac{1}{2}$.

   (a) Given that Z_n, conditional on Z_{n-1}, is independent of $Z_{n-2}, Z_{n-3}, \ldots, Z_1$ find $E[Z_n \mid Z_{n-1}]$ for each n so that the martingale condition is satisfied.

   (b) Show that $\mathbb{P}(\sup_{n\geq 1} Z_n \geq 1) = \frac{1}{2} \neq 0 = \mathbb{P}(\bigcup_{n\geq 1} \{Z_n \geq 1\})$

   (c) Show that for all $\epsilon > 0$, $\mathbb{P}(\sup_{n\geq 1} Z_n \geq a) \leq \frac{\mathbb{E}[Z_1]}{a - \epsilon}$.

6. Let Y be a Lazy random walk on $\mathbb{Z}^d$ starting at $y \in \mathbb{Z}^d$ and $X : \mathbb{N}_0 \to \mathbb{Z}^d$. Let Z_i, $i \in \mathbb{N}_0$, be i.i.d. Bernoulli random variables with mean q.

   (a) Define the stopping time

$$\tau := \min\{i \geq 0 : Y(i) = X(i), Z_i = 1\}.$$

   Show that

$$\mathbb{P}_y^Y(\tau \leq n) = 1 - \mathbb{E}_y^Y\left[(1-q)^{\sum_{i=0}^{n} 1_{\{Y(i)=X(i)\}}}\right].$$

(b) If we let $p_n^Y(y)$ denote the n-step transition probability kernel of Y, then show that

$$\begin{aligned} p_n^Y(0) &\geq p_n^Y(y), \\ p_n^Y(0) &\geq p_{n+1}^Y(0) \end{aligned}$$

for all $n \geq 0,\ y \in \mathbb{Z}^d$.

7. Let $\{N_y\}_{y\in\mathbb{Z}^d}$ be i.i.d. Poisson random variables with mean ν and let $S(\cdot)$ be a simple random walk starting at 0 and $\{Y_j^y(\cdot)\}_{y\in\mathbb{Z}^d, 1\leq j\leq N_y}$ be a collection of independent random walks having the same distribution as $S(\cdot)+y$. Let

$$\xi(n,x) := \sum_{y\in\mathbb{Z}^d, 1\leq j\leq N_y} \delta_x(Y_j^y(n)).$$

Let $X : \mathbb{N}\cup\{0\} \to \mathbb{Z}^d$ with $X(0) = 0$. Show that

$$\sigma^X(n) = \mathbb{E}^\xi\left[(1-q)^{\sum_{i=0}^n \xi(i,X(i))}\right] = \exp\Big\{-\nu\sum_{y\in\mathbb{Z}^d} w^{q,X}(n,y)\Big\},$$

where

$$w^{q,X}(n,y) := 1 - \mathbb{E}_y^Y\left[(1-q)^{\sum_{i=0}^n 1_{\{Y(i)=X(i)\}}}\right],$$

and under $\mathbb{E}_y^Y$ the random walk Y has the same distribution as $S(\cdot)+y$.