Homework Set 6
Due Date: March 31st, 2023
Problems Due: 2,3

1. Let (X_i) be i.i.d. $\mathbb{R}$–valued random variables such that

$$\varphi(t) = \mathbb{E}\, e^{tX_1} < \infty \qquad \forall t \in \mathbb{R} \tag{1}$$

and $\mathbb{E}[X_1] < 0$

   (a) Show that $\mathbb{P}(S_n \geq 0) \leq \rho^n$ where $\rho = \inf_{t\in\mathbb{R}} \varphi(t)$.

   (b) Show that there is a unique $\tau > 0$ such that

$$\varphi(\tau) = \rho, \quad \varphi'(\tau) = 0.$$

   (c) *Exponential Tilting:* Let $F(x) = \mathbb{P}(X_1 \leq x)$ be the distribution function of X_1. Define the function $\widehat{F}$, is called the *Cramér-transform* of F, by

$$\widehat{F}(x) = \frac{1}{\rho} \int_{(-\infty,x]} e^{\tau y} dF(y).$$

      i. Show that $\widehat{F}$ is a distribution function.
      ii. Let $(\widehat{X}_i)$ be an i.i.d. sequence of random variables with distribution function $\widehat{F}$. Then $\mathbb{E}[\widehat{X}_1] = 0$ and $\mathrm{var}[\widehat{X}_1] = \widehat{\sigma}^2 \in (0, \infty)$.
      iii. Let $\widehat{S}_n = \sum_{i=1}^n \widehat{X}_i$. Then $\mathbb{P}(S_n \geq 0) = \rho^n \mathbb{E}\big(e^{-\tau \widehat{S}_n} 1_{\{\widehat{S}_n \geq 0\}}\big)$.
      iv. Using the central limit theorem show that $\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left(e^{-\tau \widehat{S}_n} 1_{\{\widehat{S}_n \geq 0\}}\right) \geq 0$.
      v. Conclude that $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geq 0) = \log \rho$.

2. Assume the following version of:

   Cramer's Theorem: Let (X_i) be i.i.d. $\mathbb{R}$–valued random variables such that

$$0 \in \text{interior}\{t \in \mathbb{R} : \varphi(t) = \mathbb{E}\, e^{tX_1} < \infty\} \tag{2}$$

   Let $S_n = \sum_{i=1}^n X_i$. Then for all $a > \mathbb{E}X_1$

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geq an) = -I(a),$$

   where

$$I(z) = \sup_{t\in\mathbb{R}} [zt - \log \varphi(t)].$$

   (a) Assume $\mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 1) = \frac{1}{2}$.

      i. Show that

$$I(z) = \begin{cases} \log 2 + z \log z + (1-z)\log(1-z) & z \in [0,1] \\ \infty & \text{otherwise.} \end{cases} \tag{3}$$

      ii. Show that I is strictly convex inside $[0,1]$, and has a unique zero at $z = \frac{1}{2}$
      iii. Show that

$$\sum_{n\geq 1} \mathbb{P}(|\tfrac{1}{n} S_n - \tfrac{1}{2}| > \delta) < \infty \qquad \forall\ \delta > 0, \tag{4}$$

iv. Deduce the Strong law of large numbers of S_n via the Borel-Cantelli Lemma.

(b) Find I: when $X_i \sim$

i. X with $\mathbb{P}(X = a) = 1$ for some $a \in \mathbb{R}$.

ii. Exponential (α), for some $\alpha > 0$

iii. Poisson(α), for some $\alpha > 0$

iv. Normal(α,1), for some $\alpha > 0$

v. X where $\mathbb{P}(X = 1) = \mathbb{P}(X = 2) = \mathbb{P}(X = 3) = \frac{1}{3}$.

3. With the same hypothesis prove a similar theorem for $\mathbb{P}(S_n \leq an)$ and $a < \mathbb{E}X_1$, with the same formula for I.