Homework Set 5
Due Date: March 24th, 2023

Discrete Time Fourier Transform for finite sequences. Let $\{X_t : t = 0, 1, 2, \ldots, N-1\}$ be a set of real(or complex) numbers (sometimes called finite sequence). Define its orthonormal discrete Fourier transform (ODFT) to be the sequence

$$F_k = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} X_t e^{-i\frac{2\pi tk}{N}}, \ k = 0, \ldots, N-1. \tag{1}$$

1. Suppose $X_t \in \mathbb{R}$ for all $t = 0, 1, 2, \ldots, N-1$.
   (a) Show that F_0 is real-valued.
   (b) Show that $F_{N-k} = F_k^*$ for $1 \leq k < \frac{N}{2}$.
   (c) If N is even then $F_{\frac{N}{2}}$ is real-valued when N is even.
2. Let $\mathcal{F}$ be the $N \times N$ martix whose (k,t)th element be given by

$$\frac{1}{\sqrt{N}} \exp(-i\frac{2\pi tk}{N}),$$

   where $0 \leq k, t \leq N-1$. Let $\mathbf{X}$ be the column vector formed with entries of X and $\mathbf{F}$ be the the column vector formed with the entries of F.
   (a) Verify that $\mathbf{F} = \mathcal{F}\mathbf{X}$.
   (b) Verify that $\mathcal{F}$ is a unitary matrix. That is

$$\mathcal{F}^*\mathcal{F} = I_N.$$

   (c) Conclude that $\mathbf{X} = \mathcal{F}^*\mathbf{F}$ and write an expression as in (1) for X in terms of F.

Discrete Time Fourier Transform for infinite sequences. Let $\{a_t : t \in \mathbb{Z}\}$ be an infinite sequence of real (or complex numbers) such that

$$\sum_{t=-\infty}^{\infty} |\ a_t\ |^2 < \infty.$$

Then the Discrete Time Fourier Transform (DTFT) of a is given by

$$A(f) \equiv \sum_{t=-\infty}^{\infty} a_t e^{-i2\pi ft}, \qquad -\infty < f < \infty$$

1. Show that:
   (a) $A(\cdot)$ is periodic with unit period. That is, show that

$$A(f + j) = A(f), \text{ for all } j \in \mathbb{Z}.$$

   (b) if a_t is real-valued, then $A(-f) = A^*(f)$

(c) if a_t is periodic with period N then observe that

$$A(f) = A(f + 2\pi N).$$

Conclude that it is enough to sum over one period of a to work with the Discrete Fourier transform (as for finite sequences) with

$$A_k = \sum_{t=0}^{N-1} a_t e^{ik\frac{2\pi}{N}t}.$$

2. Compute the discrete-time Fourier transform of the following sequence:

(a) $a_t = \begin{cases} \frac{1}{4^t} & t \geq 0, t \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$

(b) $a_t = \begin{cases} 1 & t \in \{0, 1, 2, 3, 4\} \\ 0 & \text{otherwise} \end{cases}$

3. Suppose b_t is another time series, and

$$a * b_t = \sum_{m \in \mathbb{Z}} a_m b_{t-m}$$

(a) Show that for $t \in \mathbb{Z}$,

$$a_t = \int_{-1/2}^{1/2} A(f) e^{i2\pi ft}\, df$$

(b) Show that

$$\sum_{t=-\infty}^{\infty} a * b_t e^{-i2\pi ft} = A(f)B(f)$$

4. (High Pass Filter) Suppose

$$a_t = \begin{cases} 1/2, & \text{t=0;} \\ -1/4, & t = \pm 1; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } b_t = \frac{3}{16}\left(\frac{4}{5}\right)^{|t|} + \frac{1}{20}\left(-\frac{4}{5}\right)^{|t|}, t \in \mathbb{Z}$$

(a) Find $A(f)$ and $B(f)$

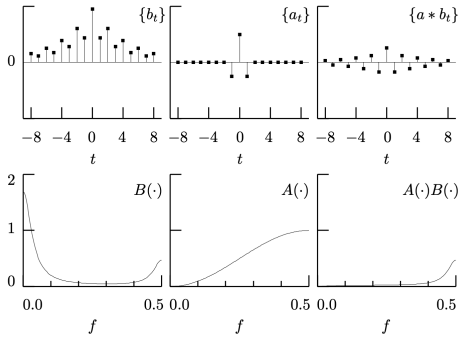
(b) Plot $A(f)B(f)$. Compare them with the figures given below and assess their significance.

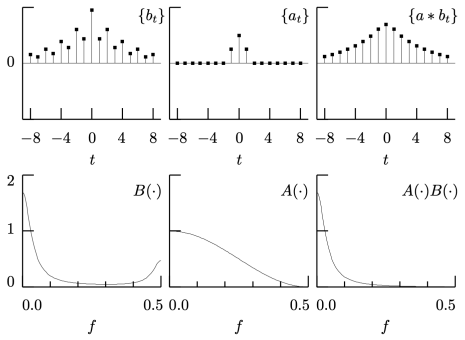
5. (Low Pass Filter) Suppose

$$a_t = \begin{cases} 1/2, & \text{t=0;} \\ 1/4, & t = \pm 1; \\ 0, & \text{otherwise.} \end{cases} \quad \text{and } b_t = \frac{3}{16}\left(\frac{4}{5}\right)^{|t|} + \frac{1}{20}\left(-\frac{4}{5}\right)^{|t|}, t \in \mathbb{Z}$$

(a) Find $A(f)$ and $B(f)$

(b) Plot $A(f)B(f)$. Compare them with the figures given below and assess their significance.





6. Let $N > 0, \{a_t : t \in \mathbb{Z}\}$ be such that $a_t = 0$ for $t \notin \{0, 1, 2, \ldots N-1\}$. Let $\{A(f) : -\frac{1}{2} < f < \frac{1}{2}\}$ be its discrete Fourier transform. We generate the periodic sequence,

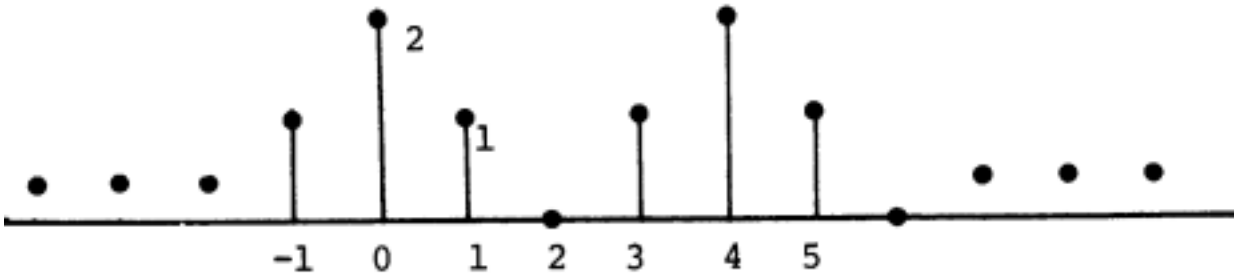
$$y_t = \sum_{r=-\infty}^{\infty} a_{t+rN}.$$

   (a) Find the period of y.

   (b) Write an expression for the discrete Fourier transform $\{B_k : K = 0, 1, \ldots N-1\}$ in terms of a.

   (c) Write an expression relating B to A.

7. The below graph describes the period 3, sequence $\{\tilde{x}_n\}_{n\in\mathbb{Z}}$,



   Determine the Fourier Coefficients $\tilde{X}_k : k = 0, 1, 2, 3$.