Homework Set 4
Due Date: March 17th, 2023

1. Show that the binary tree satisfies I_∞ with $C_0 = 3$
2. Let $\delta > 0$. If (Γ, μ) satisfies $(I_{\alpha+\delta})$ then show that it satisfies (I_α).
3. Exercise 3.12 in [Bar17]
4. Exercise 3.20 in [Bar17]
5. Consider the Lamplighter group.

$$V = \mathbb{Z} \times \{0,1\}^{\mathbb{Z}} = \{(n, \{\xi_i\}_{i\in\mathbb{Z}} : n \in \mathbb{Z}, \xi_i \in \{0,1\}\}.$$

Then place an edge between $(n, \{\xi_i\}_{i\in\mathbb{Z}})$ and $(n+1, \{\xi'_i\}_{i\in\mathbb{Z}})$ if and only if $\xi_k = \xi'_k$ for $k \notin \{n, n+1\}$.

   (a) Show that $\Gamma = (V, E)$ has exponential volume growth.

   (b) Suppose X_n is a random walk on Γ with natural weights. Let $\mathbf{0} = \{0, \{0\}_{i\in\mathbb{Z}}\}$. Show that there is a c_0 such that

$$\mathbb{P}^{\mathbf{0}}(X_n = \mathbf{0}) \geq c_o n^{\frac{1}{2}} 2^{-(2n)^{\frac{1}{2}}}$$

   Hint: you may follow the argument outlined in [Bar17, Example Page 98], but please do write out the proof presenting all details.

6. Let (Γ, μ) be a weighted graph. Assume that the spectral radius

$$\rho = \limsup_{n\to\infty} (\mathcal{P}_n(x,y))^{\frac{1}{n}}$$

is well-defined (i.e. independent of $x, y \in V$)

   (a) Show that $\rho = \lim_{n\to\infty} (\mathcal{P}_{2n}(x,x))^{\frac{1}{2n}}$

   (b) Show that for any $x, y \in V$ $p_n(x,y) \leq (\mu_x \mu_y)^{-\frac{1}{2}} \rho^n$

   (c) Suppose (Γ, μ) has controlled weights with constant C_2 then show that

$$\mid B(x,n) \mid \leq C_2^n, \qquad \mu(B(x,n)) \leq 2C_2^n \mu_x$$

   (d) Suppose (Γ, μ) has controlled weights with constant C_2 and $\rho < 1$ then show hat

$$\mathbb{P}^x(\liminf_{n\to\infty} \frac{d(x, X_n)}{n} > 0) = 1$$

   Hint: you may follow the argument outlined in [Bar17, Theorem 3.46], but please do write out the proof presenting all details.

References

[Bar17] Martin T Barlow. *Random walks and heat kernels on graphs*, volume 438. Cambridge University Press, 2017.