Homework Set 3
Due Date: March 10th, 2023

1. Consider the graph $\mathbb{Z}^d, d \geq 3$ with natural weights. Show that there exists $c_1, c_2 > 0$ such that

$$\frac{c_1}{|\ x\ |^{d-2}} \leq g(0,x) \leq \frac{c_2}{|\ x\ |^{d-2}}$$

Hint: Use Local Central Limit Theorem.

2. Let $(\Gamma = (V,E), \mu)$ be a weighted graph. For any $A \subset V$, define the potential operator $G_A : C(V) \to \mathbb{R}$ by

$$G_A f(x) = \sum_{y \in V} g_A(x,y) f(y) \mu_y.$$

Find $\Delta_A G_A$ when $A \subset V$, $A \neq V$ and Γ is recurrent. Show that

$$\Delta G f = -f$$

when Γ is transient and $f \in L^1(V)$.

3. Let $\{\xi_{i,n} : 1 \leq i, n\}$ be an i.i.d. collection of random variables on $\mathbb{N} \cup \{0\}$ such that $\mathbb{E}[\xi_{i,n}] = \mu < \infty, \mathrm{Var}(\xi_{i,n}) = \sigma^2$. Suppose $X_0 = 1$ and

$$X_n = \sum_{i=1}^{X_{n-1}} \xi_{i,n}$$

for each $n \geq 1$.

(a) Compute $\mathbb{E}[X_n]$ and $\mathrm{Var}[X_n]$.

(b) Using Markov inequality show that if $\mu < 1$ then $P(X_n > 0) \to 0$ as $n \to \infty$

(c) Using the fact that

$$P(X_n > 0) \geq \frac{(\mathbb{E}[X_n])^2}{\mathbb{E}[X_n^2]}$$

show that $P(X_n \geq 1, \forall n \geq 1) > 0$ when $\mu > 1$.

(d) Show that $Z_n = \frac{X_n}{\mu^n}$ is a martingale. Can you study the behaviour of X_n at $\mu = 1$?