Homework Set 2
Due Date: February 10th, 2023

Common:

1. Let $(\Gamma = (V, E), \mu)$ be a locally finite, connected, countably infinite vertex, weighted graph. Let $\Omega = V^{\mathbb{Z}_+}$. For any $n \geq 0$, let $X_n : \Omega \to V$ be given by $X_n(\omega) = \omega_n$, For any $x \in V$ let $\mathbb{P}^x$ be the unique Probability on $(\Omega, \mathcal{F})$ such that

$$\mathbb{P}^x(X_0 = x_1, X_1 = x_2, \ldots, X_n = x_n) = 1_x(x_0) \prod_{i=1}^{n} \mathcal{P}(x_{i-1}, x_i),$$

where $x_i \in V$ and $\mathcal{P}(x, y) = \frac{\mu_{xy}}{\mu_x}$.

   (a) For $z \in V$, let $T_z = \min\{n \geq 0 : X_n = z\}$ and $T_z^+ = \min\{n \geq 0 : X_n = z\}$. Show that the following conditions are equivalent:
   (T1) $\exists x \in V$ such that $\mathbb{P}^x(T_x^+ < \infty) < 1$.
   (T2) For all $x \in V$, $\mathbb{P}^x(T_x^+ < \infty) < 1$.
   (T3) For all $x \in V$, $\sum_{n=0}^{\infty} \mathbb{P}^x(X_n = x) < \infty$.
   (T4) For all $x, y \in V$ with $x \neq y$, $\mathbb{P}^x(T_y < \infty) < 1$ or $\mathbb{P}^x(T_y < \infty) < 1$.
   (T5) For all $x, y \in V$ with $x \neq y$, $\mathbb{P}^x(\sum_{n=0}^{\infty} 1(X_n = y) < \infty) = 1$

   (b) Let $C(V) = \{f : V \to \mathbb{R}\}$. Show that $P_n : C(V) \to C(V)$ defined by

$$P_n f(x) = \sum_{n=0}^{\infty} \mathbb{P}^x(X_n = y) f(y),$$

   satisfies $P_n = (P_1)^n$

2. Let (X_i, d_i), $i = 1, 2$ be metric spaces. A map $\phi : X_1 \to X_2$ is a rough isometry if there exist positive constants C_1, C_2 such that

$$\frac{1}{C_1}(d_1(x, y) - C_2) \leq d_2(\phi(x), \phi(y)) \leq C_1(d_1(x, y) + C_2)$$

and

$$\cup_{x \in X_1}\{y \in X_2 : d_2(\phi(x), y) \leq C_2\} = X_2.$$

   If there exists a rough isometry between two spaces they are said to be roughly isometric.

   (a) Show that $\mathbb{Z}^d, \mathbb{R}^d, [0, 1] \times \mathbb{R}^d$ are roughly isometric spaces.
   (b) Let G be a finitely generated infinite group, and λ, Λ' be two sets of generators. Let Γ, Γ' be the associated Cayley graphs. Then Γ, Γ' are roughly isometric.

3. Let $(\Gamma_i, \mu_i), i = 1, 2$ be weighted graphs and have controlled weights, i.e.

$$\frac{\mu_{xy}}{\mu_x} \geq \frac{1}{C_2} \text{ whenever } x \sim y.$$

   A map $\phi : V_1 \to V_2$ is a rough isometry between (Γ_1, μ_1) and (Γ_1, μ_2) if:

(i) ϕ is a rough isometry between the metric spaces (V_1, d_{Γ_1}) and (V_2, d_{Γ_2}) (with constants C_1 and C_2).

(ii) There exists $0 < C_3 < \infty$ such that for all $x \in V_1$

$$\frac{1}{C_3}\mu_1(x) \leq \mu_2(\phi(x)) \leq C_3\mu_1(x)$$

(a) Let $\alpha > 0$. Consider the graph $\mathbb{Z}_+$ with weights $\mu^{\alpha}_{n,n+1} = \alpha^n$.

i. Show the graph $(\mathbb{Z}_+, \mu^{\alpha})$ has controlled weights. Does it have bounded weights ?

ii. Show that the graph is recurrent if and only if $\alpha \leq 1$.

iii. For $\alpha \neq \beta$ are $(\mathbb{Z}_+, \mu^{\alpha})$ and $(\mathbb{Z}_+, \mu^{\beta})$ roughly isometric ?

For Topics in Applied Stochastic Process Class

4. Let $\Omega = \{-1, +1\}^{\mathbb{N}}$ equipped with the probability denoted by $\mathbb{P}$, such that

$$\mathbb{P}(\{\omega \in \Omega : \pi_N(\omega) = \tilde{\omega}) = \frac{1}{2^N} = \mathbb{P}_N(\{\tilde{\omega}\}),$$

where $N \in \mathbb{N}$, $\tilde{\omega} \in \Omega_N = \{-1, 1\}^N$, equipped with the uniform distribution, denoted by $\mathbb{P}_N$ and $\pi_N : \Omega \to \Omega_N$ be the cannonical projection.

Consider for $k \geq 1$, $X_k : \Omega \to \{-1, 1\}$ be given by $X_k(\omega) = \omega_k$ and for $1 \leq n$, let $S_n : \Omega \to \mathbb{Z}$ be given by $S_n(\omega) = \sum_{k=1}^n X_k(\omega)$ and $S_0 = 0$.

Definition: *Let $\mathcal{A}_n$ be the events that are observable by time n. We shall say a sequence of random variables, $\{H_n\}$ is a martingale w.r.t. the filtration $\mathcal{A}_n$ if*

$$\mathbb{E}[\mid H_n \mid] < \infty \quad \textit{and} \quad \mathbb{E}[H_n \mid S_{n-1}, \ldots, S_1] = H_{n-1}$$

(a) Show that $\xi_n = S_n^2 - n$ is a martingale w.r.t the filtration $\mathcal{A}_n$.

(b) Find $\{a_n\}_{n\geq 1}$ such that $\eta_n = S_n^3 + a_n S_n$ is a martingale w.r.t filtration $\mathcal{A}_n$.

(c) Find $\{b_n\}_{n\geq 1}$ such that $\delta_n = S_n^4 + b_n S_n^2 + c_n$ is a martingale w.r.t filtration $\mathcal{A}_n$.

(d) Let $\{V_k\}_{k\geq 1}$ be a *predictable process*, that is for $c \in \mathbb{R}$,

$$\{V_k = c\} \in \mathcal{A}_{k-1}.$$

Then show that

$$Z_0 = 0 \qquad Z_n = \sum_{k=1}^{n} V_k(S_k - S_{k-1})$$

is a martingale w.r.t the filtration $\mathcal{A}_n$.

5. Let X, Y, Z be discrete random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $\mathbb{P}(Y = y) > 0$ and $\mathbb{P}(Z = z) > 0$.

(a) Show that $\mathbb{E}[XY \mid Y = y] = y\mathbb{E}[X \mid Y = y]$

(b) If X is independent of Y then $\mathbb{E}[X \mid Y = y] = \mathbb{E}[X]$ and hence $\mathbb{E}[X \mid Y] = \mathbb{E}[X]$

6. An urn contains R red and G green balls. At each time we draw a ball from the urn, then replace it, and add C balls of the colour drawn. Let $\{X_n\}_{n\geq 1}$ be the fraction of greem balls after the n-th draw. Show that X_n is a martingale.

For Topics in Discrete Probability Class

4. (For M.Math only) Show that $C_o(V) = \{f : V \to \mathbb{R} : \text{Supp}(f) \text{ is finite } \}$ is dense in $L^p(V, \mu)$ for all $p \in [1, \infty]$.

5. (For M.Math only) For all $p, r \in [1, \infty]$, let $A : L^p(V, \mu) \to L^r(V, \mu)$. Show that

$$\| A \|_{p \to q} = \sup\{\| Af \|_q : \| f \|_p \leq 1\}$$

is a norm.

6. (For M.Math only) Let $H^2(V) = \{f \in C(V) : \mathcal{E}(f, f) < \infty\}$ where

$$\mathcal{E}(f, g) = \frac{1}{2} \sum_{x \in V} \sum_{y \in V} \mu_{xy}(f(x) - f(y))(g(x) - g(y)) \text{ and } \| f \|_{H^2} = \mathcal{E}(f, f) + f(\rho)^2,$$

for $f, g \in C(V)$ and (fixed) $\rho \in V$.

   (a) Show that $H^2(V)$ is a Hilbert space.

   (b) Let $f_n \in H^2$ with $\sup_n \| f_n \|_{H^2} < \infty$. Then there exists $\{f_{n_k}\}$ and $f \in H^2$ such that for each $x \in V$,

$$\lim_{k \to \infty} f_{n_k}(x) = f(x) \text{ and } \| f \|_{H^2} \leq \liminf_{k \to \infty} \| f_{n_k} \|_{H^2}$$