Due: Friday January 20th 2021

Let $\mathcal{A}_n$ be the events that are observable by time n.Let $N \in \mathbb{N}$. Consider

$$\Omega_N = \{\omega = (\omega_1, \omega_2, \ldots, \omega_N) : \omega_i \in \{-1, +1\}$$

equipped with the uniform distribution, denoted by $\mathbb{P} \equiv \mathbb{P}_N$. For $1 \le k \le N$, let $X_K : \Omega_N \to \{-1, 1\}$ be given by $X_k(\omega) = \omega_k$ and for $1 \le n \le N$, let $S_n : \Omega_N \to \{-1, 1\}$ be given by $S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$ and $S_0 = 0$.

1. For $1 \le n \le N$, show that the mode of S_n is $\{0, 1\}$ that is
$$\max\{\mathbb{P}(S_n = a) : a \in \mathbb{Z}\} = \begin{cases} \mathbb{P}(S_{2k} = 0) & \text{if } n = 2k, k \in \mathbb{N} \\ \mathbb{P}(S_{2k-1} = 1) & \text{if } n = 2k-1, k \in \mathbb{N} \end{cases} = \binom{2k}{k}\frac{1}{2^{2k}}$$
2. For $a < b, a, b \in \mathbb{Z}$, $1 \le n \le N$ show that
$$\mathbb{P}(a \le S_n \le b) \le (b - a + 1)\mathbb{P}(S_n \in \{0, 1\})$$
and conclude that $\lim_{N\to\infty} \mathbb{P}(a \le S_N \le b) = 0$.
3. Let $-\infty < a < 0 < b < \infty, a, b \in \mathbb{Z},$,
$$\sigma_a = \min\{k \ge 1 : S_k = a\} \qquad \text{and} \qquad \sigma_b = \min\{k \ge 1 : S_k = b\}.$$
   (a) Let $\tau_N = \min\{\sigma_a, \sigma_b, N\}$. Show that τ_N is a Stopping time.
   (b) Show that
   $$\mathbb{E}(S_{\tau_N}) = a\mathbb{P}(\sigma_a < \sigma_b, \sigma_a \le N) + b\mathbb{P}(\sigma_b < \sigma_a, \sigma_b \le N) + \mathbb{E}(S_N 1(\min\{\sigma_a\sigma_b\} > N))$$
   and
   $$\mathbb{E}(\tau_N) = a^2\mathbb{P}(\sigma_a < \sigma_b, \sigma_a \le N) + b^2\mathbb{P}(\sigma_b < \sigma_a, \sigma_b \le N) + \mathbb{E}(S_N^2 1(\min\{\sigma_a, \sigma_b\} > N)).$$
   (c) Show that
   $$1 - \mathbb{P}(\sigma_a < \sigma_b, \sigma_a \le N) - \mathbb{P}(\sigma_b < \sigma_a, \sigma_b \le N) = \mathbb{P}(\min\{\sigma_a, \sigma_b\} > N)$$
   (d) Limits as $N \to \infty$.
      i. $\mathbb{P}(\min\{\sigma_a, \sigma_b\} > N) \to 0$ as $N \to \infty$.
      ii. $\mathbb{E}(S_N 1(\min\{\sigma_a, \sigma_b\} > N)) \to 0$ as $N \to \infty$.
      iii. Show that there exists $\alpha_a, \alpha_b \in [0, 1]$ such that
      $$\alpha_a = \lim_{n\to\infty} \mathbb{P}(\sigma_a < \sigma_b, \sigma_a \le N) \qquad \text{and} \qquad \alpha_b = \lim_{n\to\infty} \mathbb{P}(\sigma_b < \sigma_a, \sigma_b \le N).$$
      iv. Conclude that
      $$\alpha_a + \alpha_b = 1 \qquad \text{and } a\alpha_a + b\alpha_b = 0.$$
      Find α_a, α_b.
      v. $\mathbb{E}(\tau_N) \to -ab$ as $N \to \infty$.
   (e) Can you provide an interpretation to answers from (d)(iv) and (d) (v) ?
4. Let $\mathcal{A}_n$ be the events that are observable by time n. Show that $\mathcal{A}_n$ is closed under complimentation and intersections.
5. Let $a \in \mathbb{N}$ and $\sigma_a = \min\{k \ge 1 : S_k = a\}$. Show that
$$\mathbb{P}(\sigma_a = n) = \frac{a}{n}\mathbb{P}(S_n = a)$$