1 Local Central Limit Theorem

The following is the version of the local central limit theorem we use.

Proposition 1.1 (Local Central Limit Theorem). *Let $\{X_n\}_{n\geq 0}$ denote a simple symmetric random walk in $\mathbb{Z}^d$. There exists a constant C such that for all positive integers n and points and $x, y \in \mathbb{Z}^d$ such that n and $y-x$ have the same parity (that is, $n+|y_1-x_1|+\cdots+|y_d-x_d|$ is even) there exists $A, B > 0$ such that*

$$\left| p_n(x,y) - \frac{A}{n^{\frac{d}{2}}} \exp\left(-\frac{Bd(x,y)^2}{n}\right)\right| \leq \frac{C}{n^{\frac{d}{2}+1}} \tag{1.1}$$

Proof. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent and identically distributed random variables defined on $\mathbb{Z}^d$ with the same distribution as ξ. The probability distribution of ξ is given by

$$\mathbb{P}\{\xi = x\} = \frac{1}{2d} \qquad \text{for } x \in \{e_1, e_2, \ldots, e_d, -e_1, -e_2, \ldots, -e_d,\}$$

with e_i being the standard unit length coordinate vectors with 1 in the i-th coordinate and 0 elsewhere. Then the simple symmetric random walk, X_n, starting at a point x, (equivalently in law) is given by

$$X_n = x + \xi_1 + \xi_2 + \cdots + \xi_n \qquad \text{for all } n \geq 1,$$

with $X_0 = x$. The characteristic function for ξ is given by

$$\phi_\xi(\theta) = \mathbb{E}[\exp(i\xi\cdot\theta)] = \sum_{y\in \text{Range}(\xi)} \mathbb{P}(\xi = y)\exp(iy\cdot\theta) = \frac{1}{2d}\sum_{y\in\text{Range}(\xi)} e^{iy\cdot\theta} = \frac{1}{d}\sum_{j=1}^{d}\cos(\theta_j), \tag{1.2}$$

for $\theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$.

We will assume without loss of generality that $x = 0$, and n, y have the same parity. As the ξ_i are i.i.d., the characteristic function for X_n is given by

$$\phi_{X_n}(\theta) = [\phi_\xi(\theta)]^n = \left[\frac{1}{d}\sum_{j=1}^{d}\cos(\theta_j)\right]^n, \tag{1.3}$$

for $\theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$. We will need the following standard Fourier inversion result.

Lemma 1.2. *If $Y = (Y_1, \ldots, Y_d)$ is a $\mathbb{Z}^d$-valued random variable whose characteristic function is given by $\phi_Y(\cdot)$, then the following equation holds*

$$\mathbb{P}(Y = x) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \phi_Y(\theta)\exp\left(-i\theta\cdot x\right)d\theta. \tag{1.4}$$

We defer the proof to the Appendix in Section **??**. Using Lemma 1.2 and (1.3) we have,

$$\mathbb{P}(X_n = x) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\left[\frac{1}{d}\sum_{j=1}^{d}\cos(\theta_j)\right]^n \exp\left(i\theta\cdot x\right)d\theta. \tag{1.5}$$

It can be noticed that in the integral above, on replacing the term θ with $\theta+(\pi,\pi,\ldots,\pi)$, the absolute value of the integrand doesn't change. Hence, the integral over $A = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\times[-\pi,\pi]^{(d-1)}$, will be half of the original integral. Hence, we have that

$$\mathbb{P}(X_n = x) = \frac{2}{(2\pi)^d}\int_{A}\left[\frac{1}{d}\sum_{j=1}^{d}\cos(\theta_j)\right]^n \exp\left(i\theta\cdot x\right)d\theta. \tag{1.6}$$

Now using Taylor's Theorem for $\cos(\cdot)$ around 0 we have that

$$\cos(\eta) = 1 - \frac{\eta^2}{2} + \frac{\eta^4}{4!}\cos(\eta^\star),$$

for any $\eta \in [-\pi, \pi]$ with $\eta^\star$ being between 0 and η. Using this coordinate-wise in (1.2) we see that there exists $r \in (0, \frac{\pi}{2})$ such that for $|\theta| \le r$,

$$0 < \phi_\xi(\theta) \le 1 - \frac{1}{4d}|\theta|^2, \tag{1.7}$$

and there exists $0 < \rho \equiv \rho(r) < 1$ such that

$$| \phi_\xi(\theta) | \le \rho \qquad \text{for all } \theta \in A \cap \{|\theta| > r\}. \tag{1.8}$$

So we split the integral in (1.6) as a sum of the following two integrals as $A = (A \cap \{|\theta| \le r\}) \sqcup (A \cap \{|\theta| > r\})$.

$$\begin{aligned}
\mathbb{P}(X_n = x) &= \frac{2}{(2\pi)^d} \int_{A \cap \{|\theta| \le r\}} [\phi_X(\theta)]^n \exp(i\theta \cdot x)\, d\theta + \frac{2}{(2\pi)^d} \int_{A \cap \{|\theta| > r\}} [\phi_X(\theta)]^n \exp(i\theta \cdot x)\, d\theta. \\
&= \frac{2}{(2\pi)^d} \int_{\{|\theta| \le r\}} [\phi_\xi(\theta)]^n \exp(i\theta \cdot x)\, d\theta + \frac{2}{(2\pi)^d} \int_{A \cap \{|\theta| > r\}} [\phi_X(\theta)]^n \exp(i\theta \cdot x)\, d\theta. \\
&= \frac{2}{(2\pi\sqrt{n})^d} \int_{|\alpha| \le r\sqrt{n}} \exp\left(-ix \cdot \frac{\alpha}{\sqrt{n}}\right) \phi_\xi^n\left(\frac{\alpha}{\sqrt{n}}\right) d\alpha + \frac{2}{(2\pi)^d} \int_{A \cap \{|\theta| > r\}} [\phi_X(\theta)]^n \exp(i\theta \cdot x)\, d\theta,
\end{aligned} \tag{1.9}$$

where we did a change of variable of $\theta \to \frac{\alpha}{\sqrt{n}}$ in the first integral. To show (1.1) we will need to compare the left hand side with the Gaussian density. For this we will need to understand the regions were the $\phi_{X_n}(\cdot)$ is close to the Gaussian density. This is made precise by the following lemma:

Lemma 1.3. *Let $n \ge 1$ and $\phi_\xi(\cdot)$ be as in (1.2).For $\alpha \in \mathbb{R}^d$ such that $| \alpha | \le n^{\frac{1}{4}}$ we have*

$$| \phi_\xi^n(\frac{\alpha}{\sqrt{n}}) - \exp\left(-\frac{| \alpha |^2}{2d}\right) | \le | \alpha |^4 \exp\left(-\frac{\alpha^2}{2d}\right) k_n, \tag{1.10}$$

where $k_n \equiv k_n(\alpha)$ is such that there exists $C > 0$ such that for $| \alpha | \le \sqrt[4]{n}$,

$$| k_n | \le \frac{C}{n} \qquad \forall n \ge 1.$$

We provide a proof of this lemma in Appendix Section **??**. For proving (1.1) we will thus split the first integral in (1.9) into the region applicable for (1.10) to hold and the compliment, along with the algebra of adding and subtracting the Gaussian density. This is described as follows:

$$\mathbb{P}(X_n = x) := \frac{2}{(2\pi\sqrt{n})^d} (I_1 + I_2 + I_3 + I_4) + I_5 \tag{1.11}$$

where I_1, I_2, I_3, I_4 and I_5 are

$$\begin{aligned}
I_1 &= \int_{|\alpha| \le n^{\frac{1}{4}}} \exp\left(-\frac{ix \cdot \alpha}{\sqrt{n}}\right) \left[\phi_\xi^n\left(\frac{\alpha}{\sqrt{n}}\right) - \exp\left(-\frac{\alpha^2}{2d}\right)\right] d\alpha \\
I_2 &= \int_{\mathbb{R}^d} \exp\left(-\frac{ix \cdot \alpha}{\sqrt{n}}\right) \exp\left(-\frac{\alpha^2}{2d}\right) d\alpha \\
I_3 &= -\int_{|\alpha| \ge n^{\frac{1}{4}}} \exp\left(-\frac{ix \cdot \alpha}{\sqrt{n}}\right) \exp\left(-\frac{\alpha^2}{2d}\right) d\alpha \\
I_4 &= \int_{n^{\frac{1}{4}} \le |\alpha| \le rn^{\frac{1}{2}}} \exp\left(-ix \cdot \frac{\alpha}{\sqrt{n}}\right) \phi_X^n\left(\frac{\alpha}{\sqrt{n}}\right) d\alpha. \\
I_5 &= \frac{2}{(2\pi)^d} \int_{A \cap \{|\theta| > r\}} [\phi_X(\theta)]^n \exp(i\theta \cdot x)\, d\theta.
\end{aligned}$$

It is immediate that:

$$I_2 = (2\pi d)^{\frac{d}{2}}\, e^{-\frac{|x|^2 d}{2n}}. \tag{1.12}$$

So rearranging (1.11) we have that

$$\mathbb{P}(X_n = x) - 2\left(\frac{d}{2\pi n}\right)^{\frac{d}{2}} \exp\left(-\frac{d|x|^2}{2n}\right) = \frac{2}{(2\pi\sqrt{n})^d} (I_1 + I_3 + I_4) + I_5 \tag{1.13}$$

For I_1 we use (1.9) from Lemma 1.3, to obtain

$$\begin{aligned}
| I_1 | &\leq \int_{|\alpha| \leq n^{\frac{1}{4}}} \exp\left(-\frac{|\alpha|^2}{2d}\right) |\alpha|^4 k_n d\alpha \\
&= k_n \int_{|\alpha| \leq n^{\frac{1}{4}}} \exp\left(-\frac{|\alpha|^2}{2d}\right) |\alpha|^4 d\alpha \\
&= k_n c_1 \int_0^{n^{\frac{1}{4}}} \exp\left(-\frac{r^2}{2d}\right) r^4 r^{d-1} dr \\
&= k_n c_1 \int_0^{n^{\frac{1}{2}}} \exp\left(-\frac{t}{2d}\right) t^{(\frac{d}{2}+2)} dt \\
&\leq k_n c_2 \int_0^{\infty} \exp\left(-\frac{t}{2d}\right) t^{(\frac{d}{2}+2)} dt \\
&\leq k_n c_3 \int_0^{\infty} \left(\frac{1}{2d}\right)^{(\frac{d}{2}+2)} \frac{\exp\left(-\frac{t}{2d}\right) t^{(\frac{d}{2}+2)}}{\Gamma(\frac{d}{2}+2)} dt \\
&= c_4 k_n. \qquad (1.14)
\end{aligned}$$

For I_3, using a change of variable we have

$$\begin{aligned}
| I_3 | &\leq \int_{|\alpha| \geq n^{\frac{1}{4}}} \exp\left(-\frac{\alpha^2}{2d}\right) d\alpha = \int_{n^{\frac{1}{4}}}^{\infty} \exp\left(-\frac{r^2}{2d}\right) r^{d-1} dr \\
&\leq c_1 \int_{n^{\frac{1}{4}}}^{\infty} \exp\left(-\frac{r^2}{4d}\right) r dr \\
&= c_2 \int_{n^{\frac{1}{2}}}^{\infty} \exp\{-\frac{t}{4d}\} dt \\
&= c_3 \left[-e^{-\frac{x}{4d}}\right]_{n^{\frac{1}{2}}}^{\infty} \\
&= c_4 e^{-\frac{n^{\frac{1}{2}}}{4d}} \qquad (1.15)
\end{aligned}$$

We bound I_4 similarly. For $|\alpha| \leq r\sqrt{n}$, we have $|\frac{\alpha}{\sqrt{n}}| \leq r$. Therefore, from (1.7) we have

$$0 \leq \phi(\frac{\alpha}{\sqrt{n}}) \leq 1 - \frac{|\alpha|^2}{4dn} \leq e^{-\frac{|\alpha|^2}{4dn}}.$$

Consequently,

$$|I_4| \leq \int_{n^{\frac{1}{4}} \leq |\alpha| \leq r n^{\frac{1}{2}}} \phi^n\left(\frac{\alpha}{\sqrt{n}}\right) d\alpha \leq \int_{n^{\frac{1}{4}} \leq |\alpha| \leq r n^{\frac{1}{2}}} e^{-\frac{|\alpha|^2}{4d}} d\alpha \leq \int_{|\alpha| \geq n^{\frac{1}{4}}} e^{-\frac{|\alpha|^2}{4d}} d\alpha \leq c_5 e^{-\frac{n^{\frac{1}{2}}}{8d}}, \qquad (1.16)$$

where the last inequality follows from the same arugment as in (1.15). Finally, for I_5, using (1.8), we have

$$|I_5| \leq \frac{2}{(2\pi)^d} \int_{A \cap \{|\theta| > r\}} |\phi_\xi(\theta)|^n d\theta \leq c_1 \rho^n, \qquad (1.17)$$

where $c_1 > 0$ is bounded by a constant times the volume of the set A. From (1.14),(1.15), (1.16), and (1.17) we have that,

$$| \mathbb{P}(X_n = x) - 2\left(\frac{d}{2\pi n}\right)^{\frac{d}{2}} \exp\left(-\frac{d < |x|^2}{2n}\right) | \leq c_6[k_n + e^{-\frac{n^{\frac{1}{2}}}{4d}} + c_4 e^{-\frac{n^{\frac{1}{2}}}{8d}} + \rho^n] \qquad (1.18)$$

The result now follows as there exists $c_1, c_2 > 0$ such that

$$c_1 \mid x \mid^2 \leq d(x, 0) \leq c_2 \mid x \mid^2 .$$

□

Proof of Lemma 1.2. From the definition of the characteristic function we have

$$\phi_Y(\theta) = \sum_{y \in \mathbb{Z}^d} \mathbb{P}\{Y = y\} e^{i \cdot y\theta}.$$

For $x \in \mathbb{Z}^d$,

$$\phi_Y(\theta)e^{-i\theta\cdot x} = \sum_{y\in\mathbb{Z}^d} \mathbb{P}\{Y=y\}e^{i\theta\cdot y}e^{-i\theta\cdot x}.$$

On integrating both sides the equation over $[-\pi,\pi]^d$, we get

$$\int_{[-\pi,\pi]^d} \phi_Y(\theta)e^{-i\theta\cdot x}d\theta = \int_{[-\pi,\pi]^d} \left(\sum_{y\in\mathbb{Z}^d} \mathbb{P}(Y=y)e^{i\theta\cdot(y-x)}\right) d\theta. \tag{1.19}$$

Verifying the condition for Fubini's theorem on the right side of Equation (1.19) above we get

$$\begin{aligned}\int_{[-\pi,\pi]^d} \sum_{y\in\mathbb{Z}^d} \left|\mathbb{P}\{Y=y\}e^{i\theta\cdot(y-x)}\right| d\theta &= \int_{[-\pi,\pi]^d} \left(\sum_{y\in\mathbb{Z}^d} \mathbb{P}\{Y=y\}\right) d\theta \\ &= \int_{[-\pi,\pi]^d} d\theta = (2\pi)^d < \infty.\end{aligned}$$

Hence we can apply Fubini's Theorem to Equation (1.19) to get

$$\int_{[-\pi,\pi]^d} \phi_Y(\theta)e^{-i\theta\cdot x}d\theta = \sum_{y\in\mathbb{Z}^d} \mathbb{P}\{Y=y\} \int_{[-\pi,\pi]^d} e^{i\theta\cdot(y-x)}d\theta. \tag{1.20}$$

If $y \neq x + 2\pi k$ for any $k \in \mathbb{Z}^d$, then

$$\int_{[-\pi,\pi]^d} e^{i\theta\cdot(y-x)}d\theta = 0.$$

If $y - x = 2\pi k$ for some $k \in \mathbb{Z}^d$, the integral becomes

$$\int_{[-\pi,\pi]^d} e^{i(2\pi k)\theta}d\theta = \int_{[-\pi,\pi]^d} 1 d\theta = (2\pi)^d.$$

Substituting the above values in (1.20) and rearranging, we have:

$$\mathbb{P}\{Y=y\} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \phi_Y(\theta)e^{-i\theta\cdot y}d\theta. \tag{1.21}$$

□

Proof of Lemma 1.3. Using Taylor's Theorem for $\cos(\cdot)$ around 0 we have that

$$\cos(\eta) = 1 - \frac{\eta^2}{2} + \frac{\eta^4}{4!}\cos(\eta^\star),$$

for any $\eta \in [-\pi,\pi]$ with $\eta^\star$ being between 0 and η. Therefore

$$\phi_\xi(\frac{\alpha}{\sqrt{n}}) = \frac{1}{d}\sum_{j=1}^d \cos(\frac{\alpha_j}{\sqrt{n}}) = 1 - \frac{|\,\alpha\,|^2}{2dn} + \frac{1}{24dn^2}\sum_{j=1}^d \alpha_j^4\cos(\alpha_j^\star)$$

So, for $|\,\alpha\,| < \sqrt[4]{n}$ we have

$$\begin{aligned}\phi_\xi^n(\frac{\alpha}{\sqrt{n}}) &\left[1 - \frac{|\,\alpha\,|^2}{2dn} + \frac{1}{24dn^2}\sum_{j=1}^d \alpha_j^4\cos(\alpha_j^\star)\right]^n \\ &= \left[1 - \frac{|\,\alpha\,|^2}{2dn}\right]^n \left[1 + \frac{\frac{1}{24dn^2}\sum_{j=1}^d \alpha_j^4\cos(\alpha_j^\star)}{1 - \frac{|\alpha|^2}{2dn}}\right]^n \\ &= \left[1 - \frac{|\,\alpha\,|^2}{2dn}\right]^n \left[1 + |\,\alpha\,|^4\, b_n\right]^n \end{aligned} \tag{1.22}$$

where $b_n \equiv b_n(\alpha)$ is such that there exists $C > 0$ such that (for all $d \geq 1$)

$$| b_n | \leq \frac{C}{n^2} \qquad \forall n \geq 1.$$

Using Taylor's Theorem for exponential function we have that

$$\exp(-x) = 1 - x + \frac{x^2}{2} e^{-x^\star},$$

for any $x \in \mathbb{R}$ with $x^\star$ being between 0 and x. Now for any $n \geq 1$

$$\begin{aligned}
\exp(-\frac{| \alpha |^2}{2d}) &= \left[\exp(-\frac{\alpha^2}{2dn})\right]^n \\
&= \left[1 - \frac{| \alpha |^2}{2dn} + \frac{| \alpha |^4}{4dn^2} e^{-\alpha_n^\star}\right]^n \\
&= \left[1 - \frac{| \alpha |^2}{2dn}\right]^n \left[1 + \frac{\frac{|\alpha|^4}{4dn^2} e^{-\alpha_n^\star}}{1 - \frac{|\alpha|^2}{2dn}}\right]^n
\end{aligned}$$

with $\alpha_n^\star$ being between 0 and $\frac{\alpha^2}{2d}$. For $| \alpha | < \sqrt[4]{n}$,

$$\exp(-\frac{| \alpha |^2}{2d}) = \left[1 - \frac{| \alpha |^2}{2dn}\right]^n \left[1 + | \alpha |^4 \, d_n\right]^n \tag{1.23}$$

where $d_n \equiv d_n(\alpha)$ is such that there exists $C_1 > 0$ such that

$$| d_n | \leq \frac{C_1}{n^2} \qquad \forall n \geq 1.$$

From (1.22) and (1.23) we have

$$\begin{aligned}
| \phi_\xi^n(\frac{\alpha}{\sqrt{n}}) - \exp\left(-\frac{| \alpha |^2}{2d}\right) | &= | \left[1 - \frac{| \alpha |^2}{2dn}\right]^n \left[1 + | \alpha |^4 \, b_n\right]^n - \exp(-\frac{| \alpha |^2}{2d}) | \\
&= \exp(-\frac{| \alpha |^2}{2d}) \, | \left[\frac{1 + | \alpha |^4 \, b_n}{1 + | \alpha |^4 \, d_n}\right]^n - 1 |
\end{aligned} \tag{1.24}$$

Using Taylors Theorem for $f(\cdot) = \frac{1}{1+\cdot}$ we have

$$\frac{1}{1+a} = 1 - a + \frac{a^2}{(1+a^\star)^3}.$$

for all $a > 0$ and with $a^\star \in (0, a)$. So, for some $d_n^\star \in (0, | \alpha |^4 \, d_n)$,

$$\begin{aligned}
\frac{1}{1 + | \alpha |^4 \, d_n} &= 1 - | \alpha |^4 \, d_n + \frac{\left(| \alpha |^4 \, d_n\right)^2}{(1 + d_n^\star)^3} \\
&= 1 - | \alpha |^4 \, d_n + | \alpha |^8 \, e_n,
\end{aligned} \tag{1.25}$$

where $e_n \equiv e_n(\alpha)$ is such that there exists $C_3 > 0$ such that for $| \alpha | \leq \sqrt[4]{n}$,

$$| e_n | \leq \frac{C_3}{n^4} \qquad \forall n \geq 1.$$

Hence substituting (1.25) in (1.24) we have

$$| \phi_\xi^n(\frac{\alpha}{\sqrt{n}}) - \exp\left(-\frac{| \alpha |^2}{2d}\right) | = \exp(-\frac{| \alpha |^2}{2d}) \, | \left[(1 + | \alpha |^4 \, b_n)(1 - | \alpha |^4 \, d_n + | \alpha |^8 \, e_n)\right]^n - 1 | \tag{1.26}$$

From, (1.22), (1.23), and (1.25) we have

$$\begin{aligned}
(1 + | \alpha |^4 \, b_n)(1 - | \alpha |^4 \, d_n + | \alpha |^8 \, e_n) &= | 1 + | \alpha |^4 \, (b_n - d_n) + | \alpha |^8 \, (e_n - b_n d_n) + | \alpha |^{12} \, b_n e_n | \\
&= 1 + | \alpha |^4 \left[(b_n - d_n) + | \alpha |^4 \, (e_n - b_n d_n) + | \alpha |^8 \, b_n e_n\right] \\
&= 1 + | \alpha |^4 \, f_n
\end{aligned} \tag{1.27}$$

where $f_n \equiv f_n(\alpha)$ is such that there exists $C_4 > 0$ such that for $\mid \alpha \mid \leq \sqrt[4]{n}$,

$$\mid f_n \mid \leq \frac{C_4}{n^2} \qquad \forall n \geq 1.$$

So using (1.27) in (1.26) we have

$$\begin{aligned} \mid \phi_\xi^n(\frac{\alpha}{\sqrt{n}}) - \exp\left(-\frac{\mid \alpha \mid^2}{2d}\right) \mid &= \exp(-\frac{\mid \alpha \mid^2}{2d}) \left[\sum_{k=1}^{n} \binom{n}{k} \mid \alpha \mid^{4k} f_n^k\right] \\ &= \exp(-\frac{\mid \alpha \mid^2}{2d}) \mid \alpha \mid^4 k_n, \end{aligned} \tag{1.28}$$

where $k_n = \sum_{k=1}^{n} \binom{n}{k} \mid \alpha \mid^{4k} f_n^k$ and

$$\mid k_n \mid = \mid \sum_{k=1}^{n} \binom{n}{k} \mid \alpha \mid^{4k-4} f_n^k \mid \leq \sum_{k=1}^{n} \binom{n}{k} n^{k-1} \frac{1}{n^{2k}} = \frac{1}{n} \sum_{k=1}^{n} \binom{n}{k} \frac{1}{n^k} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k!} \leq \frac{e}{n}.$$

This completes the proof. □