

**Ground Rules:** Individual work. No referencing: class notes, books, homework question sheets, websites and social media. Okay to ONLY reference and use any pre-written (in your own hand writing) solutions of Homework 3. You may use any result shown in class or in the Homework(s) (other than the quiz-question itself).

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**0.(3 points)** Please write this statement at the beginning of your quiz solution: *I certify that I have read the Ground Rules and shall follow them. I have not consulted anyone else for the duration of the quiz.* Then write your name and signature.

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**1. (15 points)** Let  $N \in \mathbb{N}$ . Consider

$$\Omega_N = \{\omega = (\omega_1, \omega_2, \dots, \omega_N) : |\omega_i| = 1, \omega_i \in \mathbb{Z}\}$$

equipped with the uniform distribution, denoted by  $\mathbb{P} \equiv \mathbb{P}_N$ . For  $1 \leq k \leq N$ , let  $X_K : \Omega_N \rightarrow \{-1, 1\}$  be given by  $X_k(\omega) = \omega_k$  and for  $1 \leq n \leq N$ , let  $S_n : \Omega_N \rightarrow \{-1, 1\}$  be given by  $S_n(\omega) = \sum_{k=1}^n X_k(\omega)$  and  $S_0 = 0$ .

For  $a < b, a, b \in \mathbb{Z}, 1 \leq n \leq N$  show that

$$\mathbb{P}(a \leq S_n \leq b) \leq (b - a + 1)\mathbb{P}(S_n \in \{0, 1\})$$

and conclude that  $\lim_{N \rightarrow \infty} \mathbb{P}(a \leq S_N \leq b) = 0$ .

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**2.(2 points)** Please submit your answer as a single p.d.f. file on the Moodle platform.

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Solution:

let  $N \geq 1$ . Problem 1 HW 4  $\Rightarrow$  for  $1 \leq n \leq N$

$$\max \{ P(S_n = a) : a \in \mathbb{Z} \} = \begin{cases} P(S_{2k} = 0) & \text{if } n = 2k \\ P(S_{2k-1} = 0) & \text{if } n = 2k-1 \end{cases}$$

As  $P(S_{2k-1} = 0) = 0 = P(S_{2k} = 1)$ , for  $k \geq 1$ ,  
the above implies

$$1 \leq n \leq N, \max \{ P(S_n = a) : a \in \mathbb{Z} \} \leq P(S_n \in \{0, 1\}) \quad \text{--- } \textcircled{+}$$

Now, for  $1 \leq n \leq N$ , we have from  $\textcircled{+}$

$$\begin{aligned} P(a \leq S_n \leq b) &= \sum_{k=a}^b P(S_n = k) \\ &\leq (b-a+1) P(S_n \in \{0, 1\}) \quad \text{--- } \textcircled{*} \end{aligned}$$

Problem 1 HW 4  $\Rightarrow$

$$P(S_N \in \{0, 1\}) = \binom{2k}{k} \frac{1}{2^{2k}}$$

if  $N = 2k$  or  $N = 2k-1$

}  $\textcircled{xx}$

# STIRLING'S Formula :-

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1$$

$\therefore$  let  $\delta = \frac{1}{2}$ . Then  $\exists N_0$  :

$$\left| \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} - 1 \right| < \delta \quad \forall n \geq N_0$$

$$\text{i.e. } \frac{(n^n e^{-n} \sqrt{2\pi n})}{2} \leq n! \leq \frac{3}{2} n^n e^{-n} \sqrt{2\pi n} \quad \forall n \geq N_0$$

$\therefore$  using the above we have for  $k \geq N_0$

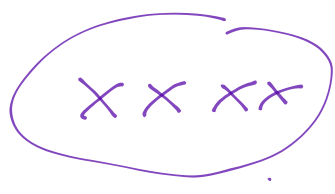
$$\begin{aligned} \binom{2k}{k} \frac{1}{2^{2k}} &= \frac{(2k)!}{(k!)^2} \frac{1}{2^{2k}} \\ &\leq \frac{\frac{3}{2} (2k)^{2k} e^{-2k} \sqrt{2\pi} \sqrt{2k}}{\frac{1}{2} (k^k e^{-k} \sqrt{2\pi} \sqrt{k})^2} \cdot \frac{1}{2^{2k}} \end{aligned}$$



$$\therefore \binom{2k}{k} \frac{1}{2^{2k}} \leq \frac{3}{\sqrt{\pi}} \frac{1}{\sqrt{k}} \quad \forall k \geq N_0$$

— (xxx)

Observe that  $\forall k \geq 1$

If  $N \geq 2k-1$  or  $N = 2k$

then  $\frac{1}{\sqrt{k}} \leq \frac{\sqrt{2}}{\sqrt{N}}$  — 

Then , , , , implies that  
 $\forall N \geq N_0$

$$0 \leq \mathbb{P}(a \leq S_N \leq b) \leq \frac{3\sqrt{2}}{\sqrt{\pi}} \frac{1}{\sqrt{N}}$$

Let  $\varepsilon > 0$  be given.

$$\exists N_1 \geq N_0 \text{ s.t. } \frac{1}{\sqrt{N}} < \frac{\sqrt{\pi}}{3\sqrt{2}} \cdot \varepsilon, \forall N \geq N_1$$

$\therefore \forall n \geq N_1$  we have

$$0 \leq \mathbb{P}(a \leq S_N \leq b) < \varepsilon$$

As  $\varepsilon > 0$  was arbitrary, we have that

$$\lim_{N \rightarrow \infty} \mathbb{P}(a \leq S_N \leq b) = 0$$

□