Ground Rules: Individual work. No referencing: class notes, books, homework question sheets, websites and social media. Okay to ONLY reference and use any pre-written (in your own hand writing) solutions of Homework 3. You may use any result shown in class or in the Homework(s) (other than the quiz-question itself).

0.(3 points) Please write this statement at the begining of your quiz solution: *I certify that I have read the Ground Rules and shall follow them. I have not consulted anyone else for the duration of the quiz.* Then write your name and signature.

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1. (15 points) Let $N \in \mathbb{N}$. Consider

$$\Omega_N = \{\omega = (\omega_1, \omega_2, \dots, \omega_N) :| \omega_i |= 1, \omega_i \in \mathbb{Z}\}$$

equipped with the uniform distribution, denoted by $\mathbb{P} \equiv \mathbb{P}_N$. For $1 \leq k \leq N$, let $X_K : \Omega_N \to \{-1, 1\}$ be given by $X_k(\omega) = \omega_k$ and for $1 \leq n \leq N$, let $S_n : \Omega_N \to \{-1, 1\}$ be given by $S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$ and $S_0 = 0$.

For $a < b, a, b \in \mathbb{Z}$, $1 \leq n \leq N$ show that

$$\mathbb{P}(a \leq S_n \leq b) \leq (b - a + 1)\mathbb{P}(S_n \in \{0, 1\})$$

and conclude that $\lim\limits_{N\to\infty} \mathbb{P}(a \leq S_N \leq b) = 0$.

2.(2 points) Please submit your answer as a single p.d.f. file on the Moodle platform.

Solution:

let $N \geq 1$. Problem 1 HW 4 $\Rightarrow$ for $1 \leq n \leq N$

$$\max\{\mathbb{P}(S_n = a) : a \in \mathbb{Z}\} = \begin{cases} \mathbb{P}(S_{2k} = 0) & \text{if } n = 2k \\ \mathbb{P}(S_{2k-1} = 0) & \text{if } n = 2k-1 \end{cases}.$$

As $\mathbb{P}(S_{2k-1} = 0) = 0 = \mathbb{P}(S_{2k} = 1)$, for $k \geq 1$, the above implies

$$1 \leq n \leq N, \quad \max\{\mathbb{P}(S_n = a) : a \in \mathbb{Z}\} \leq \mathbb{P}(S_n \in \{0,1\}) \quad \text{— } (+)$$

Now, for $1 \leq n \leq N$, we have from $(+)$

$$\mathbb{P}(a \leq S_n \leq b) = \sum_{k=a}^{b} \mathbb{P}(S_n = k)$$

$$\leq (b - a + 1)\, \mathbb{P}(S_n \in \{0,1\}) \quad \text{— } (*)$$

Problem 1 HW 4 $\Rightarrow$.

$$\left.\begin{array}{l} \mathbb{P}(S_N \in \{0,1\}) = \binom{2k}{k} \dfrac{1}{2^{2k}} \\ \\ \text{if } N = 2k \text{ or } N = 2k-1 \end{array}\right\} \text{— } (**)$$

STIRLING's Formula :-

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1$$

$\therefore$ let $\delta = \frac{1}{2}$. Then $\exists$ N_0 :

$$\left| \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} - 1 \right| < \delta \qquad \forall\, n \geq N_0$$

i.e. $$\frac{\left(n^n e^{-n}\sqrt{2\pi n}\right)}{2} \leq n! \leq \frac{3}{2} n^n e^{-n}\sqrt{2\pi n}$$

$\forall\, n \geq N_0$

$\therefore$ using the above we have for $k \geq N_0$

$$\binom{2k}{k}\frac{1}{2^{2k}} = \frac{2k!}{(k!)^2}\frac{1}{2^{2k}}$$

$$\leq \frac{\frac{3}{2}}{\frac{1}{2}} \frac{(2k)^{2k} e^{-4k}\sqrt{2\pi}\sqrt{2k}}{\left(k^k e^{-2k}\sqrt{2\pi}\sqrt{k}\right)^2} \cdot \frac{1}{2^{2k}}$$

$$\therefore \binom{2k}{k}\frac{1}{2^{2k}} \leq \frac{3}{\sqrt{\pi}}\frac{1}{\sqrt{k}} \qquad \forall\, k \geq N_0$$

$\times\times\times$

Observe that $\forall\ k \geq 1$

If $N = 2k-1$ or $N = 2k$

then $\frac{1}{\sqrt{k}} \leq \frac{\sqrt{2}}{\sqrt{N}}$ — $(\times\times\times\times)$

Then $(\times)$, $(\times\times)$, $(\times\times\times)$, $(\times\times\times\times)$, implies that $\forall\ N \geq N_0$

$$0 \leq \mathbb{P}(a \leq S_N \leq b) \leq \frac{3\sqrt{2}}{\sqrt{\pi}} \frac{1}{\sqrt{N}}$$

Let $\varepsilon > 0$ be given.

$\exists\ N_1 \geq N_0$ s.t. $\frac{1}{\sqrt{N}} < \frac{\sqrt{\pi}}{3\sqrt{2}} \cdot \varepsilon$, $\forall\ N \geq N_1$

$\therefore\ \forall\ n \geq N_1$ we have

$$0 \leq \mathbb{P}(a \leq S_N \leq b) < \varepsilon$$

As $\varepsilon > 0$ was arbitrary, we have that

$$\lim_{N \to \infty} \mathbb{P}(a \leq S_N \leq b) = 0 \qquad \square$$