Due: Thursday April 15, 2021, 10pm
Problems to be turned in: 2,3

1. Let $\{X_k\}_{k\geq 1}$ be i.i.d. X with $\mathbb{P}(X=1)=\frac{1}{2}=\mathbb{P}(X=-1)$. Suppose $S_0=0, S_n=\sum_{i=1}^n X_i$ and let

$$J=\min\{n\geq 1: S_n=-a\} \text{ for } a\in\mathbb{N}\cup\{0\}$$

be a stopping time with respect to the filtration $\mathcal{A}_n$ of observable events with regard to $\{X_i\}_{i\geq 1}$

  (a) For $\theta\in\mathbb{R}, n\geq 1$, let $Z_n=\frac{\exp(\theta S_n)}{(\cosh(\theta))^n}$. Show that Z_n is a martingale.

  (b) Show that, when $\theta<0$, the martingale $Z_{\min\{n,J\}}$ is a bounded martingale.

  (c) Show[1] that for $0<\alpha<1$, one has

$$\mathbb{E}[\alpha^J]=\left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right)^a.$$

  (d) Let $a=1$, calculate $\mathbb{P}(J=n)$ for all $n\in\mathbb{N}$.

2. Let $\{X_k\}_{k\geq 1}$ be a sequence of i.i.d. X such that $p=\mathbb{P}(X=1)=1-\mathbb{P}(X=-1)$ with $0\leq p<\frac{1}{2}$. Let $S_n=\sum_{i=1}^n X_i$.

  (a) Show that $\mathbb{P}(\cup_{n\geq 1}\{S_n\geq k\})=\mathbb{P}(\cup_{n\geq 1}\{S_n\geq 1\})^k$

  (b) Let $x=\mathbb{P}(\cup_{n\geq 1}\{S_n\geq 1\})$. Show that x solves $x=p+(1-p)x^2$ and find x

  (c) Show that $\frac{p}{1-p}=\exp(-r^\star)$ where $r^\star$ is the unique positive root of $\mathbb{E}[e^{rX} \; = 1$. Thus conclude that the optimised Chernoff bound is satisfied with equality.

3. (Galton-Watson Process) For $n\geq 1$, let $\{X_k^n\}_{k\geq 1}$ be i.i.d. X with range of X being in $\mathbb{N}\cup\{0\}$ and $m=\mathbb{E}[X]$. Define a sequence Z_n by, $Z_1=1$ and for $n\geq 2$,

$$Z_n=\sum_{i=1}^{Z_{n-1}} X_i^n,$$

with the sum being interpreted as 0 if $Z_{n-1}=0$.

  (a) Show that $\frac{Z_n}{m^n}$ is a martingale.

  (b) Show that $P(Z_n>0)\leq m^n$ and conclude that if $\mu<1$ then that the Galton-Watson Process dies out, i.e $\mathbb{P}(Z_n>0$ infinitely often$)=0$.

[1] You may use the Bounded convergence theorem: namely if $\{X_n\}_{n\geq 1}$ are a sequence of bounded random variables and $X_n\to X$ with probability 1 then $\mathbb{E}[X_n]\to\mathbb{E}[X]$

Book-Keeping Exercises

1. Soha the gambler with an initial finite capital of $r > 0$ rupees starts to play on the Siva rupee slot machine. At each play, either her rupee is lost or is returned with some additional number of rupees. Let X_i be her change of capital on the ith play. Assume that $\{X_k : i \geq 1\}$ is i.i.d. X with X taking on integer values $\{-1, 0, 1, \ldots\}$. Assume that $\mathbb{E}[X] < 0$. Soha plays until losing all her money (i.e., the initial r rupees plus subsequent winnings).

   (a) Let J be the number of plays until Soha loses all her money. Is the weak law of large numbers sufficient to argue that $\mathbb{P}(J < \infty) = 1$ or is the strong law necessary?

   (b) Assume that X is bounded and using Chernoff bound show that $\mathbb{E}[J] < \infty$. In this case find $\mathbb{E}[J]$ using Wald's identity.

2. Let $\{X_k\}_{k\geq 1}$ be a sequence of i.i.d. $\exp(\lambda)$ and let $S_n = \sum_{i=1}^n X_i$.

   (a) Show that for all $a > \frac{1}{\lambda}$, the optimal Chernoff bound is given by

   $$P(S_n \geq na) \leq (a\lambda)^n e^{-n(a\lambda-1)}.$$

   (b) Show that $\mathbb{P}(S_n \geq na) = \sum_{i=0}^{n-1} \frac{(na\lambda)^i e^{-na\lambda}}{i!}$.

   (c) Use Stirling's formula to show that

   $$\frac{(a\lambda)^n e^{-n(a\lambda-1)}}{\sqrt{2\pi n}(a\lambda \exp(\frac{1}{12n})} \leq P(S_n \geq na) \leq \frac{(a\lambda)^n e^{-n(a\lambda-1)}}{\sqrt{2\pi n}(a\lambda - 1)}$$

3. *(Discrete Fubini Theorem-1)* Suppose for $n \geq 1$, $f_n, f : X \to \mathbb{R}$ and $f_n \to f$ uniformly on a set K in a metric space (X, d). Let x be a limit point of K. Suppose

   $$\lim_{t\to x} f_n(t) = A_n$$

   for all $n \geq 1$. Then

   (a) $\lim_{n\to\infty} A_n = A$ for some $A \in \mathbb{R}$.

   (b) $\lim_{t\to x} f(t) = A$.

   (c) $\lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t)$

   For solution see Walter Rudin, Principle of Mathematical Analysis: Theorem 7.11 Page 149.

4. *(Discrete Fubini Theorem-2)* Suppose for $m, n \geq 1$, $\{a_{mn}\}$ be a sequence of real numbers.

   $$\text{If} \qquad \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} | a_{mk} | < \infty \qquad \text{then} \qquad \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} a_{mk} = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} a_{km}.$$

   For solution see Joel Feldman Notes at : http://www.math.ubc.ca/ feldman/m321/twosum.pdf