**Due: Thursday April 8, 2021, 10pm**
*Problems to be turned in: 2,4*

1. Consider a martingale where $Z_n$ can take on only the values $2^{-n-1}$ and $1 - 2 - n - 1$, each with probability $\frac{1}{2}$.
   (a) Given that $Z_n$, conditional on $Z_{n-1}$, is independent of $Z_{n-2}, Z_{n-3}, \ldots, Z_1$ find $E[Z_n \mid Z_{n-1}]$ for each $n$ so that the martingale condition is satisfied.
   (b) Show that $\mathbb{P}(\sup_{n\geq 1} Z_n \geq 1) = \dfrac{1}{2} \neq 0 = \mathbb{P}(\bigcup_{n\geq 1}\{Z_n \geq 1\})$
   (c) Show that for all $\epsilon > 0$, $\mathbb{P}(\sup_{n\geq 1} Z_n \geq a) \leq \dfrac{E[Z_1]}{a-\epsilon}$.

2. Let $Y$ be an random walk on $\mathbb{Z}^d$ starting at $y \in \mathbb{Z}^d$ and $X : \mathbb{N}_0 \to \mathbb{Z}^d$. Let $Z_i$, $i \in \mathbb{N}_0$, be i.i.d. Bernoulli random variables with mean $q$. Define the stopping time

$$\tau := \min\{i \geq 0 : Y(i) = X(i), Z_i = 1\}.$$

   Show that

$$\mathbb{P}^Y_y(\tau \leq n) = 1 - \mathbb{E}^Y_y\Big[(1-q)^{\sum_{i=0}^n 1_{\{Y(i)=X(i)\}}}\Big].$$

3. Let $\{N_y\}_{y\in\mathbb{Z}^d}$ be i.i.d. Poisson random variables with mean $\nu$ and let $S(\cdot)$ be a simple random walk starting at 0 and $\{Y^y_j(\cdot)\}_{y\in\mathbb{Z}^d, 1\leq j\leq N_y}$ be a collection of independent random walks having the same distrbution as $S(\cdot) + y$. Let

$$\xi(n,x) := \sum_{y\in\mathbb{Z}^d, 1\leq j\leq N_y} \delta_x(Y^y_j(n)).$$

   Let $X : \mathbb{N}\cup\{0\} \to \mathbb{Z}^d$ with $X(0) = 0$. Show that

$$\sigma^X(n) = \mathbb{E}^\xi\Big[(1-q)^{\sum_{i=0}^n \xi(i,X(i))}\Big] = \exp\Big\{-\nu\sum_{y\in\mathbb{Z}^d} w^{q,X}(n,y)\Big\},$$

   where

$$w^{q,X}(n,y) := 1 - \mathbb{E}^Y_y\Big[(1-q)^{\sum_{i=0}^n 1_{\{Y(i)=X(i)\}}}\Big],$$

   and under $\mathbb{E}^Y_y$ the random walk $Y$ has the same distribution as $S(\cdot) + y$.

4. Let $X$ be a random variable such that $\mathbb{P}(X = 1) = \frac{1}{4}$ and $\mathbb{P}(X = -1) = \frac{3}{4}$.
   (a) Let $\phi(t) = \mathbb{E}[e^{tX}]$ for all $t \in \mathbb{R}$. Show that there is $\tau > 0$ such that

$$\inf_{t\in\mathbb{R}} \phi(t) = \rho, \qquad \phi(\tau) = \rho, \qquad \phi'(\tau) = 0$$

   (b) Let $X_i$ for $i \geq 1$ be i.i.d. $X$ and $S_n = \sum_{i=1}^n X_i$ with $S_0 = 0$. Using Chebychev inequality, show that

$$\limsup_{n\to\infty} \frac{1}{n}\log\mathbb{P}(S_n \geq 0) \leq \log(\rho)$$

   (c) Let $\hat{X}$ be a random variable with distribution function given by the Cramer transform of $X$. Namely

$$\mathbb{P}(\hat{X} \leq x) := \frac{1}{\rho}\mathbb{E}[e^{\tau X} 1(X \leq x)]$$

i. Show that $\mathbb{E}[\hat{X}] = 0$ and $\text{Var}[\hat{X}] = 1$.

ii. Let $\hat{S}_n = \sum_{i=1}^n \hat{X}_i$ then for any $C > 0$,

$$\mathbb{P}(0 \leq \frac{\hat{S}_n}{\sqrt{n}} \leq C) \leq \frac{\mathbb{E}[e^{-\tau \hat{S}_n} 1(\hat{S}_n \geq 0)]}{e^{-\tau C \sqrt{n}}}$$

iii. Show that $\mathbb{P}(S_n \geq 0) = \rho^n \mathbb{E}[e^{-\tau \hat{S}_n} 1(\hat{S}_n \geq 0)]$

iv. Conclude that

$$\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(S_n \geq 0) \geq \log(\rho)$$

(d) Discuss (b) and (c) with the perpsective of Large Deviations for $S_n$.

(e) *Extra Credit:* Can you adapt the above proof to show that for any $a > -\frac{1}{2}$

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(S_n \geq an) = \inf_{t \in \mathbb{R}} \{at - \log(\phi(t))\}$$

5. Let $\{X_k\}_{k \geq 1}$ be i.i.d Bernoulli $(\frac{1}{2})$ and $S_n = \sum_{k=1}^n X_k$. For $a < \frac{1}{2}$, find $\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(S_n \leq an)$