2. Let $V = \mathbb{Z}^2$ and $E = \{\{\mathbf{i},\mathbf{j}\} : \mathbf{i},\mathbf{j} \in V \text{ and } \| \mathbf{i}-\mathbf{j} \| \leq \sqrt{2}\}$ with $\| \cdot \|$ denoting the euclidean norm. Consider the standard weight function $\mu \equiv 1$ on E. Let $\{X_n\}_{n\geq 1}$ be the random walk on this weighted graph and $\mathcal{A}_n$ be the events observable by time n w.r.t $\{X_k\}_{k\geq 1}$.

(a) Show that $M_n = \| X_n \|^2 - \frac{3}{2}n$ is a martingale w.r.t the filtration $\mathcal{A}_n$.

(b) Let $\tau_R = \min\{n \geq 0 : \| X_n \|^2 \geq R^2\}$. Show that

$$\frac{2}{3}R^2 \leq E[\tau_R] \leq \frac{2}{3}(R+\sqrt{2})^2$$

Solution 2(a) : We show that $\mathbb{E}[|M_n|] < \infty$. Indeed ,$M_n = \|X_n\|^2 - \frac{3}{2}n$, but $\|X_n\| \leq \sqrt{2}n$ since the furthest the random walk can travel in n steps away from the origin, is $\sqrt{2}n$. Therefore, we have

$$\mathbb{E}[|M_n|] \leq \mathbb{E}[\|X_n\|^2] + \frac{3}{2}n \leq 2n^2 + \frac{3}{2}n < \infty$$

for all n. Consider the set

$$S = \{(1,0),(-1,0),(0,1),(0,-1),(1,1),(1,1),(1,1),(-1,-1)\}$$

and $\{\xi_k : k \geq 1\}$ i.i.d ξ such that the distribution of ξ is Uniform(S). Let $x_1,...,x_{n-1} = (x,y) \in \mathbb{Z}^2$ be such that $\mathbb{P}[X_1 = x_1,\ldots,X_{n-1} = x_{n-1}] > 0$. Using $X_n = \sum_{k=1}^n \xi_k$. We have

$$\begin{aligned}
\mathbb{E}[\|X_n\|^2|X_{n-1} = x_{n-1},...,X_1 = x_1] &= \mathbb{E}[\|X_{n-1}+\xi_n\|^2|X_{n-1} = x_{n-1},...,X_1 = x_1] \\
&= \mathbb{E}[\|x_{n-1}+\xi_n\|^2|X_{n-1} = x_{n-1},...,X_1 = x_1] \\
\text{(using independence)} &= \mathbb{E}[\|x_{n-1}+\xi_n\|^2] \\
&= \frac{1}{8}\sum_{z\in S}\|x_{n-1}+z\|^2 \\
&= \|x_{n-1}\|^2 + \frac{3}{2}
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}[M_n|X_{n-1},...,X_1] &= \mathbb{E}[\|X_n\|^2|X_{n-1} = x_{n-1},...,X_1 = x_1] - \frac{3}{2}n \\
&= \|x_{n-1}\|^2 + \frac{3}{2} - \frac{3}{2}n \\
&= \|x_{n-1}\|^2 + \frac{3(n-1)}{2}.
\end{aligned}$$

where we have used linearity of conditional expectation and the fact that the conditional expectation of a constant is itself in the first line. Thus M_n is a martingale.

□

Solution 2(b) : Since it is not specified in the question, we shall assume $X_0 = 0$. For any n we have

$$\{\tau_R = n\} = \{\|X_1\| < R,\ldots,\|X_{n-1}\| < R, \|X_n\| \geq R\}$$

is an event observable by time n and therefore τ_R is a stopping time for all R. We will apply the Optional Sampling Theorem for which we have to verify the two hypothesis:

$$\mathbb{E}(M_n|\tau_R > n)\mathbb{P}(\tau_R > n) \to 0 \qquad \text{and} \qquad \mathbb{E}(\|M_{\tau_R}\|^2) < \infty.$$

Let $\beta = \max_{\|x\|<R} \mathbb{P}(\tau_R > R | X_0 = x)$. It is easy to see that $\beta < 1$, for example by considering a path that goes only left or only right at each point for R steps has positive probability, depending upon the parities of the coordinates (and will result in $\tau_R < 1$). Then using the Markov property, we have that

$$\mathbb{P}(\tau_R > kR) < \beta^k \qquad \text{and consequently} \qquad \mathbb{E}[\tau_R] < \infty \qquad \text{(why ?)}.$$

Also,

$$R^2 \leq \|X_{\tau_R}\|^2 \leq \|X_{\tau_R - 1}\|^2 + 2 \leq R^2 + 2 \leq (R + \sqrt{2})^2. \tag{1}$$

Therefore

$$\mathbb{E}[| M_{\tau_R} | \leq (R + \sqrt{2})^2 + \mathbb{E}[\tau_R] + 1 < \infty$$

Second,

$$\mathbb{E}(M_n | \tau_R > n)\mathbb{P}(\tau_R > n) \leq \left(2n^2 + \frac{3}{2}n\right) \beta^{\lfloor \frac{n}{R} \rfloor} \to 0 \qquad \text{as } n \to \infty.$$

So the Optional Sampling Theorem implies that

$$\mathbb{E}[M_{\tau_R}] = \mathbb{E}[M_1] = 0 \qquad \Longleftrightarrow \qquad \mathbb{E}[\|X_{\tau_R}\|^2] = \frac{3}{2}\mathbb{E}[\tau_R].$$

Using (1) we have

$$\frac{2}{3}R^2 \leq \mathbb{E}[\tau_R] \leq \frac{2}{3}(R + \sqrt{2})^2$$

as desired.

□