Due: Thursday April 1, 2021, 10pm
Problems to be turned in: 2

1. We place M marbles in P pots. At each time unit we choose one of the marbles uniformly at random and place it in one of the urns also uniformly chosen at random. Denote by M_n to be the number of marbles in the first urn at time n. Find a_n, b_n, so that $a_n M_n + b_n$ is a martingale.

2. Let $V = \mathbb{Z}^2$ and $E = \{\{\mathbf{i}, \mathbf{j}\} : \mathbf{i}, \mathbf{j} \in V \text{ and } \parallel \mathbf{i} - \mathbf{j} \parallel \leq \sqrt{2}\}$ with $\parallel \cdot \parallel$ denoting the euclidean norm. Consider the standard weight function $\mu \equiv 1$ on E. Let $\{X_n\}_{n\geq 1}$ be the random walk on this weighted graph and $\mathcal{A}_n$ be the events observable by time n w.r.t $\{X_k\}_{k\geq 1}$.
   (a) Show that $M_n = \parallel X_n \parallel^2 - \frac{3}{2}n$ is a martingale w.r.t the filtration $\mathcal{A}_n$.
   (b) Let $\tau_R = \min\{n \geq 0 : \parallel X_n \parallel^2 \geq R^2\}$. Show that
   $$\frac{2}{3}R^2 \leq E[\tau_R] \leq \frac{2}{3}(R + \sqrt{2})^2$$

3. Let X_n be a Markov chain on S with transition matrix P and initial distribution μ. Assume S is finite and $\mathcal{A}_n$ are observable events by time n w.r.t $\{X_k\}_{k\geq 1}$. Find conditions on a and $f : S \to \mathbb{R}$ such that $M_n = a^{-n} f(X_n)$ is a martingale w.r.t the filtration $\mathcal{A}_n$.

Book-Keeping Exercises

1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space on which $\{M_n\}_{n\geq 1}$ be a martingale w.r.t to the filtration $\mathcal{A}_n$ of observable events by time n. Let $T : \Omega \to \mathbb{N}$ be a stopping such that $E[T] < \infty$. Assume that M_n are $\mathbb{Z}$ valued random variables. Let $\epsilon > 0$ be given.
   (a) Show that there is a $M > 0$ such that $\sum_{k=M}^{\infty} \mathbb{P}(\mid M_1 \mid > k) < \epsilon$.
   (b) Show that there is a $N > 0$ such that $\mathbb{P}(T > N) < \epsilon$.

   Conclude that $\lim_{n\to} \mathbb{E}[\mid M_1 \mid 1(T > N)] = 0$

2. Let $\{Z_n\}_{n\geq 1}$ be a martingale, and for some integer m, let $Y_n = Z_{n+m} - Z_m$.
   (a) Show that $\mathbb{E}[Y_n \mid Z_{n+m-1} = z_{n_m-1}, Z_{n+m-2} = z_{n_m-2}, \ldots, Z_1 = z_1] = z_{n+m-1} - z_m$
   (b) Show that $\mathbb{E}[Y_n \mid Y_{n+m-1} = y_{n_m-1}, Y_{n+m-2} = y_{n_m-2}, \ldots, Y_1 = y_1] = y_{n-1}$
   (c) Show that $\mathbb{E}[\mid Y_n \mid] < \infty$ and $\{Y_n\}_{n\geq 1}$ is a martingale.

3. Let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$. Let $Z_1 = -2$ and
   $$Z_{n+1} = Z_n(1 + X_n \left(\frac{3n+1}{n+1}\right))$$
   Let $J = \min\{n \geq 2 : \text{sign}(Z_n) = \text{sign}(Z_{n-1})\}$ be a stopping time.
   (a) Show that $\{Z_n\}_{n\geq 1}$ is a martingale.
   (b) Using induction on n show that
   $$\mathbb{P}(Z_n = -\frac{2^n}{n} \mid J > n) = 1 \qquad \text{and } \mathbb{P}(Z_n = -\frac{2^n(n-2)}{n^2 - n} \mid J = n) = 1$$
   (c) Show that $\mathbb{E} \mid Z_J \mid < \infty$ and $\lim_{n\to\infty} \mathbb{E}[Z_n \mid J > n]\mathbb{P}(J > n) = 0$