**Due: Thursday March 25th, 2020, 10pm**
*Problems to be turned in: 2*

1. An urn contains $R$ red and $G$ green balls. At each time we draw a ball from the urn, then replace it, and add $C$ balls of the colour drawn. Let $\{X_n\}_{n\geq 1}$ be the fraction of greem balls after the $n$-th draw. Show that $X_n$ is a martingale.

2. Let $\Omega = \{-1,1\}^{\mathbb{N}}$. Consider for $k \geq 1$, $X_k : \Omega \to \{-1,1\}$ which are i.i.d. $X$ with distribution given by

$$\mathbb{P}(X = 1) = p = 1 - \mathbb{P}(X = -1),$$

for some $0 < p < 1$.

   (a) For $n \geq 1$, let $S_n : \Omega \to \mathbb{Z}$ be given by $S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$ and $S_0 = 0$. Show that

$$W_n = \left(\frac{1-p}{p}\right)^{S_n}$$

   is a martingale.

   (b) Let $p = \frac{1}{2}$ and for $n \geq 1$ let

$$Y_n = \frac{1}{2} + \sum_{k=1}^{n} \frac{1}{2^{k+1}} X_k.$$

   Show that $Y_n$ is a martingale. (*Any guess if $Y_n$ converges as $n \to \infty$ and if it does what is its limiting distribution ?* )

3. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a Probability space on which $\{M_n\}_{n\geq 1}$ be a martingale w.r.t to the filtration $\mathcal{A}_n$ of observable events by time $n$. Let $V_k : \Omega \to \mathbb{R}$ be sequence of discrete random variables that are predictable, i.e.

$$\{V_k = c\} \in \mathcal{A}_{k-1}, \qquad c \in \text{Range}(V_k), k \geq 1.$$

Show that

$$M_n^V = \sum_{k=1}^{n} V_k(M_k - M_{k-1}),$$

is a martingale.

## Book Keeping Exercises

1. *Coupon Collector Problem* Let $V = \{1, 2, \dots, n\}$ and $E = \{\{i,j\} : i \in V, j \in V\}$. Consider the standard weight function $\mu \equiv 1$ on $E$. Let $\{X_n\}_{n\geq 1}$ be the random walk on this weighted graph.

   (a) Let $\tau_0^n = 0$ and let $\tau_k^n$ be the first time the walk has visited $k$ -distinct vertices. Show that $\tau_k^n - \tau_{k-1}^n$ for $1 \leq k \leq n$ are independent of each other and find their distribution.

   (b) Consider the cover time of the graph to be given by

$$\tau_{\text{cover}}^n = \tau_n^n = \max_{j=1}^{n} T^{\{j\}}$$

      i. Find $\mathbb{E}(\tau_{\text{cover}}^n)$

ii. Find $\mathrm{Var}(\tau^n_{\text{cover}})$

iii. Show that $\frac{\tau^n_{\text{cover}}}{n\log(n)} \xrightarrow{p} 1$.

(c) Why the name Coupon collector problem for this question ?

2. Let $p, q > 0$ such that $p + q = 1$. Let $r > 0$ be given. Let $d < 1 + r < u$. Let $\Omega_i = \{H, T\}$ and $\Omega = \prod_{i=1}^N \Omega_i$. Let $\mathbb{P}$ be probability on $\Omega$, such that

$$\mathbb{P}(\{(\omega_1, \dots, \omega_N)\}) = p^{\#\{j:\omega_j=H\}} q^{\#\{j:\omega_j=T\}}.$$

Let $\mathbb{E}$ denote the expectation under $\mathbb{P}$. For $1 \le i \le N$, sefine the random variables $\xi_i : \Omega_i \to \{u, d\}$, such that

$$\xi_i(H) = u \text{ and } \xi_i(T) = d.$$

Let $\mathcal{A}_n$ be the collection of events observable by time $n$. We can then define for $1 \le k \le N$,

$$S_k(\omega) = S_k((\omega_1, \dots \omega_k)) = \left(\prod_{i=1}^k \xi_i(\omega_i)\right) S_0$$

Assume $N = 4$

(a) Show that $S_3(\{H, H, H, H\}) = S_3(\{H, H, H, T\})$. Give an intuitive explanation why this is true by sketching the binomial tree of possibilities for $S_k, 1 \le k \le 4$.

(b) Show that $\mathbb{E}(S_k \mid \mathcal{A}_{k-1}) = (pu + qd)S_{k-1}$, for $k = 2, 3, 4$.

(c) Let $S_0 = 5$ be the stock price at time 0. Find $\mathbb{E}(S_k \mid \mathcal{A}_0), k = 1, 2, 3, 4$.

### Random Questions (from student discussions[1])

1. $\Omega_n$ denote the set of all permuatations $\omega : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$. Suppose we choose an element, $\omega$, uniformly from $\Omega_n$ and denote by $L_n$ the number of cycles in $\omega$. Find $\mathbb{E}(L_n)$.

2. Let $\{S^{(i)}_n\}_{n\ge1, 1\le i\le K}$ be a collection of independent simple random walks on $\mathbb{Z}$. Assume

$$S^{(1)}_0 < S^{(2)}_0 < \dots S^{(i)}_0 < S^{(i+1)}_0 < \dots < S^{(K)}_0.$$

Fix $N \in \mathbb{N}$, what is the probability that the walks $\{S^{(i)}_n\}_{n\ge1, 1\le i\le K}$ don't intersect before time $N$ and

$$S^{(i)}_N = y_i \text{ for } 1 \le i \le K \text{ and } y_j < y_{j+1}, 1 \le j \le K - 1.$$

3. Construct an explicit polynomial approximation to the function $f : [-1, 1] \to \mathbb{R}$ such that $f(x) =\mid x \mid$ i.e. For every $\epsilon > 0$ find a polynomial $p_n : [-1, 1] \to \mathbb{R}$ such that

$$\sup_{x\in[-1,1]} \mid p_n(x) - \mid x \mid \mid < \epsilon.$$

4. Construct a $C^\infty$-function $f : \mathbb{R} \to [0, 1]$ such that

$$f(x) = \begin{cases} 1 & \text{for } x \in [-1, 1] \\ 0 & \text{for } x \notin (-2, 2) \end{cases}$$

5. Six players play a game using a deck of 20 cards. 10 cards are numbered one and 10 cards are numbered two. Each player is given three cards. If a player does not received at least one card with number two, that player is eliminated from the game. Find the expected number of players that survive the first round of the game.

[1]These are questions students asking me in class or during discussions outside class.