2. Let $\Omega = \{-1, +1\}^{\mathbb{N}}$ equipped with the probability denoted by $\mathbb{P}$, such that

$$\mathbb{P}(\{\omega \in \Omega : \pi_N(\omega) = \tilde{\omega}) = \frac{1}{2^N} = \mathbb{P}_N(\{\tilde{\omega}\}),$$

where $N \in \mathbb{N}$, $\tilde{\omega} \in \Omega_N = \{-1, 1\}^N$, equipped with the uniform distribution, denoted by $\mathbb{P}_N$ and $\pi_N : \Omega \to \Omega_N$ be the cannonical projection.

Consider for $k \geq 1$, $X_k : \Omega \to \{-1, 1\}$ be given by $X_k(\omega) = \omega_k$ and for $1 \leq n$, let $S_n : \Omega \to \mathbb{Z}$ be given by $S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$ and $S_0 = 0$.

Definition: *Let $\mathcal{A}_n$ be the events that are observable by time n. We shall say a sequence of random variables, $\{H_n\}$ is a martingale w.r.t. the filtration $\mathcal{A}_n$ if*

$$\mathbb{E}[\mid H_n \mid] < \infty \quad \textit{and} \quad \mathbb{E}[H_n \mid S_{n-1}, \ldots, S_1] = H_{n-1}$$

(a) Show that $\xi_n = S_n^2 - n$ is a martingale w.r.t the filtration $\mathcal{A}_n$.

(d) Let $\{V_k\}_{k\geq 1}$ be a predictable process, that is for $c \in \mathbb{R}$,

$$\{V_k = c\} \in \mathcal{A}_{k-1}.$$

Then show that

$$Z_0 = 0 \qquad Z_n = \sum_{k=1}^{n} V_k(S_k - S_{k-1})$$

is a martingale w.r.t the filtration $\mathcal{A}_n$.

Solution 2(a) : We must prove that ξ_n is a martingale with respect to $\mathcal{A}_n$. For this , we note that

$$\mathbb{E}[|\xi_n|] \leq \mathbb{E}[|S_n|^2] + n \leq n^2 + n < \infty$$

for all n where the second inequality follows from the fact that $|S_n|^2$ is bounded above by n^2. For the second part, fix $s_{n-1}, s_{n-2}, ..., s_1$ such that $\mathbb{P}[S_{n-1} = s_{n-1}, S_{n-2} = s_{n-2}, ..., s_1 = s_1] \neq 0$. Then the conditional probability and expectation with respect to this event is well defined and we have :

$$\begin{aligned}
\mathbb{E}[\xi_n | S_{n-1} = s_{n-1}, ..., S_1 = s_1] &= \sum_{k\in\mathbb{Z}} k\mathbb{P}[\xi_n = k | S_{n-1} = s_{n-1}, ..., S_1 = s_1] \\
&= \sum_{k\in\mathbb{Z}} k\mathbb{P}[S_n^2 = n + k | S_{n-1} = s_{n-1}, ..., S_1 = s_1] \\
&= \sum_{k\in\mathbb{Z}} k\mathbb{P}[(s_{n-1} + X_n)^2 = n + k | S_{n-1} = s_{n-1}, ..., S_1 = s_1] \\
&= \sum_{k\in\mathbb{Z}} k \frac{\mathbb{P}[(s_{n-1} + X_n)^2 = n + k, S_{n-1} = s_{n-1}, ..., S_1 = s_1]}{\mathbb{P}[S_{n-1} = s_{n-1}, ..., S_1 = s_1]}
\end{aligned}$$

Since X_n is independent of $S_1, ..., S_{n-1}$, we get :

$$\begin{aligned}\sum_{k\in\mathbb{Z}} k\frac{\mathbb{P}[(s_{n-1}+X_n)^2 = n+k, S_{n-1} = s_{n-1}, ..., S_1 = s_1]}{\mathbb{P}[S_{n-1} = s_{n-1}, ..., S_1 = s_1]} &= \sum_{k\in\mathbb{Z}} k\frac{\mathbb{P}[(s_{n-1}+X_n)^2 = n+k]\mathbb{P}[S_{n-1} = s_{n-1}, ..., S_1 = s_1]}{\mathbb{P}[S_{n-1} = s_{n-1}, ..., S_1 = s_1]} \\ &= \sum_{k\in\mathbb{Z}} k\mathbb{P}[(s_{n-1}+X_n)^2 = n+k] \\ &= ((s_{n-1}+1)^2 - n)\mathbb{P}[X_n = 1] + ((s_{n-1}-1)^2 - n)\mathbb{P}[X_n = -1] \\ &= \frac{(s_{n-1}-1)^2 - n + (s_{n-1}+1)^2 - n)}{2} = s_{n-1}^2 - (n-1) = \xi_{n-1}\end{aligned}$$

as desired. Thus , ξ_n is a martingale w.r.t $\mathcal{A}_n$. □

Solution 2(d) : First note that $S_k - S_{k-1} = X_k$ therefore we have $Z_n = \sum_{k=1}^{n} V_k X_k$. Therefore :

$$\mathbb{E}[|Z_n|] \leq \sum_{k=1}^{n} \mathbb{E}[|V_k||X_k|] \leq \sum_{k=1}^{n} \mathbb{E}[|V_k|] < \infty$$

As the V_k are bounded random variables, being functions on a finite domain. For the second condition, fix n. We know that $\{V_k = c\} \in A_{k-1} \subset A_{n-1}$ for all $k \leq n$. In particular, this implies that

$$\{V_k = c\} = \{(X_1, ..., X_{n-1}) \in A'\}$$

for some set $A' \subset \{\pm 1\}^{n-1}$. But then, $X_j = S_j - S_{j-1}$, therefore $(X_1, ..., X_{n-1}) = g(S_1, ..., S_{n-1})$ for some function g, and it follows that

$$\{V_k = c\} = \{(S_1, ..., S_{n-1}) \in g^{-1}(A')\}$$

for any k. In particular, for any set $B \subset D_1 \times ... \times D_k$ where D_i denote the set of values which V_i takes for $1 \leq i \leq n$, we have

$$\{(V_1, ..., V_k) \in B\} = \{S_1, ..., S_{n-1} \in S'\}$$

for some $S' \subset \mathbb{Z}^n$. This establishes the following : there exists a function f' with appropriate domain and codomain such that $V_n = f'(S_1, ..., S_{n-1})$. Using a similar argument for other $k \leq n$ this also implies that $Z_{n-1} = \sum_{k=1}^{n-1} V_k X_k = g(S_1, ..., S_n)$ for some function g.

Let $s_{n-1}, s_{n-2}, ..., s_1$ be such that $\mathbb{P}[S_{n-1} = s_{n-1}, ..., S_1 = s_1] > 0$. Then we have, from definitions and from the independence of X_n from $S_1, ..., S_{n-1}$, in an argument analogous to $2(a)$:

$$\begin{aligned}\mathbb{E}[Z_n|S_{n-1} = s_{n-1}, ..., S_1 = s_1] &= \sum_K K\mathbb{P}[Z_n = K|S_{n-1} = s_{n-1}, ..., S_1 = s_1] \\ &= \sum_K K\mathbb{P}[Z_{n-1} + V_n X_n = K|S_{n-1} = s_{n-1}, ..., S_1 = s_1] \\ &= \sum_K K\mathbb{P}[g(s_{n-1}, ..., s_1) + X_n f'(s_{n-1}, ..., s_1) = K|S_{n-1} = s_{n-1}, ..., S_1 = s_1] \\ &= \sum_K K\frac{\mathbb{P}[g(s_{n-1}, ..., s_1) + X_n f'(s_{n-1}, ..., s_1) = K, S_{n-1} = s_{n-1}, ..., S_1 = s_1]}{\mathbb{P}[S_{n-1} = s_{n-1}, ..., S_1 = s_1]} \\ &= \sum_K K\frac{\mathbb{P}[g(s_{n-1}, ..., s_1) + X_n f'(s_{n-1}, ..., s_1) = K]\mathbb{P}[S_{n-1} = s_{n-1}, ..., S_1 = s_1]}{\mathbb{P}[S_{n-1} = s_{n-1}, ..., S_1 = s_1]} \\ &= \sum_K K\mathbb{P}[g(s_{n-1}, ..., s_1) + X_n f'(s_{n-1}, ..., s_1) = K] \\ &= [g(s_{n-1}, ..., s_1) + f'(s_{n-1}, ..., s_1)]\mathbb{P}[X_n = 1] + [g(s_{n-1}, ..., s_1) - f'(s_{n-1}, ..., s_1)]\mathbb{P}[X_n = -1] \\ &= \frac{[g(s_{n-1}, ..., s_1) + f'(s_{n-1}, ..., s_1)] + [g(s_{n-1}, ..., s_1) - f'(s_{n-1}, ..., s_1)]}{2} \\ &= g(s_{n-1}, ..., s_1) = Z_{n-1}\end{aligned}$$

which shows that Z_n is a martingale. $\square$ 3. Let X, Y, Z be discrete random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $\mathbb{P}(Y = y) > 0$ and $\mathbb{P}(Z = z) > 0$.

(a) Show that $\mathbb{E}[XY \mid Y = y] = y\mathbb{E}[X \mid Y = y]$

Solution 3(a) : As $\mathbb{P}[Y = y] > 0$, the expression $E[XY|Y = y]$ is well defined. By definition, we have :

$$E[XY|Y = y] = \sum_{k\in\mathbb{Z}} k\mathbb{P}[XY = k|Y = y]$$

we now distinguish $y = 0$ and $y \neq 0$. Indeed, if $y = 0$ then we have :

$$\sum_{k\in\mathbb{Z}} k\mathbb{P}[XY = k|Y = 0] = \sum_{k\in\mathbb{Z}} k\mathbb{P}[k = 0|Y = 0] = 0 = yE[X|Y = y]$$

trivially. For $y \neq 0$, we have :

$$\sum_{k\in\mathbb{Z}} k\mathbb{P}[XY = k|Y = y] = \sum_{k\in\mathbb{Z}} k\mathbb{P}\left[X = \frac{k}{y}\middle|Y = y\right] = y\sum_{k\in\mathbb{Z}} \frac{k}{y}\mathbb{P}\left[X = \frac{k}{y}\middle|Y = y\right]$$

We claim that

$$\sum_{k\in\mathbb{Z}} \frac{k}{y}\mathbb{P}\left[X = \frac{k}{y}\middle|Y = y\right] = \sum_{l\in\mathbb{Z}} l\mathbb{P}[X = l|Y = y]$$

To prove this, note that if $y \nmid k$ then $\mathbb{P}\left[X = \frac{k}{y}\right] = 0$ and hence $\mathbb{P}\left[X = \frac{k}{y}\middle|Y = y\right] = 0$. Therefore, the only terms that remain on the LHS are those for which $y|k$. Reindexing with $l = \frac{y}{k}$ gives us the RHS. With this, we get :

$$y\sum_{k\in\mathbb{Z}} \frac{k}{y}\mathbb{P}\left[X = \frac{k}{y}\middle|Y = y\right] = y\sum_{l\in\mathbb{Z}} l\mathbb{P}[X = l|Y = y] = yE[X|Y = y]$$

as desired. $\square$