Due: Thursday March 18th, 2020, 10pm
Problems to be turned in: 2 (a), 2 (d), 3(a)

1. Please prepare an at least one page summary notes to an at most six page notes of your class talk. The notes should at the miminmum contain:
   (a) Title of your class.
   (b) Main objective (Statement of Theorem(s) or Question(s)- including notation required to understand the result.)
   (c) An outline of the talk

   or it can be as complete a set of notes for your entire talk.

2. Let $\Omega = \{-1, +1\}^{\mathbb{N}}$ equipped with the probability denoted by $\mathbb{P}$, such that

$$\mathbb{P}(\{\omega \in \Omega : \pi_N(\omega) = \tilde{\omega}) = \frac{1}{2^N} = \mathbb{P}_N(\{\tilde{\omega}\}),$$

   where $N \in \mathbb{N}$, $\tilde{\omega} \in \Omega_N = \{-1, 1\}^N$, equipped with the uniform distribution, denoted by $\mathbb{P}_N$ and $\pi_N : \Omega \to \Omega_N$ be the cannonical projection.

   Consider for $k \geq 1$, $X_k : \Omega \to \{-1, 1\}$ be given by $X_k(\omega) = \omega_k$ and for $1 \leq n$, let $S_n : \Omega \to \mathbb{Z}$ be given by $S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$ and $S_0 = 0$.

   Definition: *Let $\mathcal{A}_n$ be the events that are observable by time n. We shall say a sequence of random variables, $\{H_n\}$ is a martingale w.r.t. the filtration $\mathcal{A}_n$ if*

$$\mathbb{E}[\mid H_n \mid] < \infty \quad \textit{and} \quad \mathbb{E}[H_n \mid S_{n-1}, \ldots, S_1] = H_{n-1}$$

   (a) Show that $\xi_n = S_n^2 - n$ is a martingale w.r.t the filtration $\mathcal{A}_n$.
   (b) Find $\{a_n\}_{n\geq 1}$ such that $\eta_n = S_n^3 + a_n S_n$ is a martingale w.r.t filtration $\mathcal{A}_n$.
   (c) Find $\{b_n\}_{n\geq 1}$ such that $\delta_n = S_n^4 + b_n S_n^2 + c_n$ is a martingale w.r.t filtration $\mathcal{A}_n$.
   (d) Let $\{V_k\}_{k\geq 1}$ be a *predictable process*, that is for $c \in \mathbb{R}$,

$$\{V_k = c\} \in \mathcal{A}_{k-1}.$$

   Then show that

$$Z_0 = 0 \qquad Z_n = \sum_{k=1}^{n} V_k(S_k - S_{k-1})$$

   is a martingale w.r.t the filtration $\mathcal{A}_n$.

3. Let X, Y, Z be discrete random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $\mathbb{P}(Y = y) > 0$ and $\mathbb{P}(Z = z) > 0$.
   (a) Show that $\mathbb{E}[XY \mid Y = y] = y\mathbb{E}[X \mid Y = y]$
   (b) If X is independent of Y then $\mathbb{E}[X \mid Y = y] = \mathbb{E}[X]$ and hence $\mathbb{E}[X \mid Y] = \mathbb{E}[X]$

Book Keeping Exercises

1. Let X_n be an irreducible Markov chain on a finite state space S with transition matrix P. For $i \in S$, a directed spanning tree on S with root at i is by definition a graph g which satisfies the following conditions:

   (I) g has vertices indexed by S, and is a tree, meaning that it is connected and has no cycles; or equivalently, there is a unique path between any two vertices in g

   (II) the edges of g are directed in such a way that for any $j \in S$, all the edges in the unique path from j to i is directed towards i - i.e., the path from j to i is directed towards i.

   Let $G(i)$ be the set of all directed spanning trees on S with root at i. For every $i \in S$, define

   $$\pi_0(i) = \sum_{g \in G(i)} \prod_{(j \to k) \in g} p_{jk},$$

   where $j \to k$ denotes any directed edge in the tree g.

   (a) Show that

   $$\pi_0(i) = \sum_{j \in S} \pi_0(j) p_{ji}, \forall i \in S,$$

   which is equivalent to

   $$\left(\sum_{g \in G(i)} \prod_{(j \to k) \in g} p_{jk}\right) \sum_{j \neq i} p_{ij} = \sum_{j \neq i} \left(\sum_{g \in G(j)} \prod_{(k \to j) \in g} p_{kj}\right) p_{ji} \tag{1}$$

   (b) Show that both sides of (1) are equal to $\sum_{g \in L} \prod_{(k \to j) \in g} p_{jk}$, where L is the set of all graphs satisfying:

   (i) every point $i \in S$ has only one directed edge pointing way from i to $j \in S$, $j \neq i$.

   (ii) the graph has exactly one closed cycle and $i \in S$ belongs to that cycle.

   (c) Conclude that X_n has a stationary distribution.

2. Let X, Y, Z be discrete random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $\mathbb{P}(Y = y) > 0$ and $\mathbb{P}(Z = z) > 0$.

   (a) Show that $\mathbb{E}[X + Y \mid Y = y] = y + \mathbb{E}[X \mid Y = y]$

   (b) Show that $\mathbb{E}[Y \mid Y = y, Z = z] = y$

3. The number of eggs N found in nests of a certain species of turtles has a Poisson distribution with mean λ. Each egg has probability p of being viable and this event is independent from egg to egg. Find the mean and variance of the number of viable eggs per nest.

4. Let $X \sim$ Uniform $\{1, 2, \ldots, n\}$ be independent of $Y \sim$ Uniform $\{1, 2, \ldots, n\}$. Let $Z = \max(X, Y)$ and $W = \min(X, Y)$.

   (a) Find the joint distribution of (Z, W).

   (b) Fine $E[Z \mid W]$.