Due: Friday March 12th, 2020
Problems to be turned in: 3

Let $\mathcal{A}_n$ be the events that are observable by time n.Let $N \in \mathbb{N}$. Consider

$$\Omega_N = \{\omega = (\omega_1, \omega_2, \dots, \omega_N) : \omega_i \in \{-1, +1\}$$

equipped with the uniform distribution, denoted by $\mathbb{P} \equiv \mathbb{P}_N$. For $1 \leq k \leq N$, let $X_K : \Omega_N \to \{-1, 1\}$ be given by $X_k(\omega) = \omega_k$ and for $1 \leq n \leq N$, let $S_n : \Omega_N \to \{-1, 1\}$ be given by $S_n(\omega) = \sum_{k=1}^n X_k(\omega)$ and $S_0 = 0$.

1. For $1 \leq n \leq N$, show that the mode of S_n is $\{0, 1\}$ that is

$$\max\{\mathbb{P}(S_n = a) : a \in \mathbb{Z}\} = \begin{cases} \mathbb{P}(S_{2k} = 0) & \text{if } n = 2k, k \in \mathbb{N} \\ \mathbb{P}(S_{2k-1} = 1) & \text{if } n = 2k-1, k \in \mathbb{N} \end{cases} = \binom{2k}{k} \frac{1}{2^{2k}}$$

2. For $a < b, a, b \in \mathbb{Z}$, $1 \leq n \leq N$ show that

$$\mathbb{P}(a \leq S_n \leq b) \leq (b - a + 1)\mathbb{P}(S_n \in \{0, 1\})$$

and conclude that $\lim\limits_{N \to \infty} \mathbb{P}(a \leq S_N \leq b) = 0$.

3. Let $-\infty < a < 0 < b < \infty, a, b \in \mathbb{Z},,$

$$\sigma_a = \min\{k \geq 1 : S_k = a\} \qquad \text{and} \qquad \sigma_b = \min\{k \geq 1 : S_k = b\}.$$

(a) Let $\tau_N = \min\{\sigma_a, \sigma_b, N\}$. Show that τ_N is a Stopping time.

(b) Show that

$$\mathbb{E}(S_{\tau_N}) = a\mathbb{P}(\sigma_a < \sigma_b, \sigma_a \leq N) + b\mathbb{P}(\sigma_b < \sigma_a, \sigma_b \leq N) + \mathbb{E}(S_N 1(\min\{\sigma_a \sigma_b\} > N))$$

and

$$\mathbb{E}(\tau_N) = a^2\mathbb{P}(\sigma_a < \sigma_b, \sigma_a \leq N) + b^2\mathbb{P}(\sigma_b < \sigma_a, \sigma_b \leq N) + \mathbb{E}(S_N^2 1(\min\{\sigma_a, \sigma_b\} > N)).$$

(c) Show that

$$1 - \mathbb{P}(\sigma_a < \sigma_b, \sigma_a \leq N) - \mathbb{P}(\sigma_b < \sigma_a, \sigma_b \leq N) = \mathbb{P}(\min\{\sigma_a, \sigma_b\} > N)$$

(d) Limits as $N \to \infty$.

i. $\mathbb{P}(\min\{\sigma_a, \sigma_b\} > N) \to 0$ as $N \to \infty$.

ii. $\mathbb{E}(S_N 1(\min\{\sigma_a, \sigma_b\} > N)) \to 0$ as $N \to \infty$.

iii. Show that there exists $\alpha_a, \alpha_b \in [0, 1]$ such that

$$\alpha_a = \lim_{n \to \infty} \mathbb{P}(\sigma_a < \sigma_b, \sigma_a \leq N) \qquad \text{and} \qquad \alpha_b = \lim_{n \to \infty} \mathbb{P}(\sigma_b < \sigma_a, \sigma_b \leq N).$$

iv. Conclude that

$$\alpha_a + \alpha_b = 1 \qquad \text{and } a\alpha_a + b\alpha_b = 0.$$

Find α_a, α_b.

v. $\mathbb{E}(\tau_N) \to -ab$ as $N \to \infty$.

(e) Can you provide an interpretation to answers from (d)(iv) and (d) (v) ?

Book Keeping Exercises

1. Show that $\{X_i\}_{1\leq i\leq N}$ are i.i.d. with distribution $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$.

2. Prove that an increment $S_m - S_k$ for $0 < k < m \leq N$ has the same distribution as S_{m-k}.

3. Prove that S_n is a Makov Chain

4. Let for $0 < k \leq N$, $a \in \mathbb{Z}, \mathbb{P}(S_k = a) > 0$. Prove that for $0 < k < m \leq N$,

$$\mathbb{P}(S_m = b \mid S_k = a) = \mathbb{P}(S_{m-k} = b - a)$$

for $b \in \mathbb{Z}$.

5. Let $\mathcal{A}_n$ be the events that are observable by time n. Show that $\mathcal{A}_n$ is closed under complimentation and intersections.

6. Let $a \in \mathbb{N}$ and $\sigma_a = \min\{k \geq 1 : S_k = a\}$. Show that

$$\mathbb{P}(\sigma_a = n) = \frac{1}{2}[\mathbb{P}(S_{n-1} = a - 1) = -\mathbb{P}(S_{n-1} = a + 1)]$$

7. *Gambler's Ruin Revisited Using Markov Chain* Let X_n be a Markov chain on S with transition matrix P and initial distribution μ.

   (a) Let $A \subset S$ and T^A be the hitting time of the set A. Let $g^A : S \to [0, 1]$ be given by

   $$g^A(i) = E_i(T^A).$$

   i. g^A is a solution of the linear system of equations given by

   $$\begin{aligned} g^A(i) &= 0 \text{ if } i \in A \\ g^A(i) &= 1 + \sum_{j\in A^c} p_{ij} g^A(j) \text{ if } i \notin A. \end{aligned} \tag{1}$$

   ii. If $u : S \to [0, \infty)$ is another solution to (1) then $u(i) \geq g^A(i)$ for all $i \in S$.

   (b) Let X_n be a Markov chain on S with transition matrix P and initial distribution μ. Let $S = \{0, 1, \ldots\}$ with $P(X_0 = 10) = 1$ and transition matrix P be given by

   $$p_{ij} = \begin{cases} 1 & \text{if } i = j = 0 \\ p & \text{if } i \geq 1, j = i + 1 \\ 1 - p & \text{if } i \geq 1, j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

   Let $A = \{0\}$. Find $g^A(10)$.