1. As discussed in class, let $\mathbb{T}_2$ be a rooted binary tree, with root ρ. For each $x \in \mathbb{T}_2$, let $\alpha(x) \in \mathbb{T}_2$ denote the ancestor of x and $\mid x \mid$ denote the distance to the root ρ. On $\mathbb{T}_2$, consider the weight function μ to be given by
$$\mu(\{x, \alpha(x)\}) = \beta^{|x|} \text{ for } x \in \mathbb{T}_2$$
where β is a positive number. Let X_n denote the canonical random walk on $(\mathbb{T}_2, \mu)$.
   (a) Show that for $\beta < \frac{1}{2}$ that the random walk X_n is recurrent.
   (b) Consider $Y_n = \mid X_n \mid$.
      i. Show that Y_n is a Markov Chain on $\mathbb{Z}_+$.
      ii. Show that Y_n is recurrent when $\beta = \frac{1}{2}$.
      iii. Show that Y_n is transient when $\beta > \frac{1}{2}$.
   (c) Conclude that X_n is transient if and only if $\beta > \frac{1}{2}$.

Solution 1(a) : Let V be the vertex set of $\mathbb{T}_2$. Let $S_n = \{x \in V : |x| = n\}$. It can be readily verified that $V = \cup_{n \geq 0} S_n$, where the union is disjoint and using an inductive argument one can show that $|S_n| = 2^n$ for all $n \geq 1$. With a slight abuse of notation, μ induces a countably additive set function on subsets of V, by

$$\text{for } x \in V, \qquad \mu(\{x\}) = \sum_{x \sim y, x,y \in V} \mu(\{x, y\}), \qquad \text{and} \qquad \mu(A) = \sum_{x \in A} \mu(\{x\}) \qquad \text{for } A \subset V.$$

We will prove that if $\beta < \frac{1}{2}$ then $\mu(V) < \infty$, which (by a result shown in class) will then imply that the graph is recurrent.

Assume $\beta < \frac{1}{2}$ Let $x \in S_n$, $n > 0$. Then, note that, x has an edge to its ancestor $\alpha(x)$ in S_{n-1} with edge weight β^n and two edges to vertices $y \in S_{n+1}$, with $\alpha(y) = x$, with edge weights β^{n+1}. This implies that $\mu(\{x\}) = \beta^n + 2\beta^{n+1}$. Therefore,

$$\mu(S_n) = \sum_{x \in S_n} \mu(\{x\}) = \sum_{x \in S_n} \beta^n + 2\beta^{n+1} = 2^n \beta^n + 2^{n+1} \beta^{n+1}$$

for $n > 1$ and $\mu(S_0) = 2\beta$.

$$\mu(V) = \lim_{m \to \infty} \sum_{n=1}^{m} \mu(S_n) = 2\beta + \lim_{m \to \infty} \sum_{n=1}^{m} ((2\beta)^n + (2\beta)^{n+1}) < \infty,$$

(since $2\beta < 1$ so the given series is a sum of two convergent geometric series). Thus, we are done.

□

Solution 1(b) (i) : X_n is a Markov chain on $\mathbb{T}_2$ with transition matrix P and initial distribution $\mathbb{P}[X_0 = \rho] = 1$, where $P = [p_{xy}]_{x,y \in \mathbb{T}_2}$ is given by

$$p_{xy} = \frac{\mu(\{x, y\})}{\mu(\{x\})} = \begin{cases} \frac{1}{1+2\beta} & \text{if } y = \alpha(x), x \in \mathbb{T}_2, x \neq \rho \\ \frac{\beta}{1+2\beta} & \text{if } x = \alpha(y), x \in \mathbb{T}_2, y \neq \rho \\ 0 & \text{otherwise.} \end{cases}$$

We will show that Y_n is a Markov chain. First, note that Y_n takes values in $\mathbb{Z}_+$ and that $Y_n = y$ if and only if $X_n \in S_y$ for any $n, y \in \mathbb{N}$. Suppose $y \in \mathbb{Z}_+$ we have $\mathbb{P}[Y_n = y] > 0$, then for some

$n \in \mathbb{N}$. Then for $y, z \in \mathbb{Z}_+$

$$\begin{aligned}
\mathbb{P}[Y_{n+1} = z | Y_n = y] &= \mathbb{P}[X_{n+1} \in S_z | X_n \in S_y] \\
&= \frac{1}{\mathbb{P}[X_n \in S_y]} \sum_{u \in S_z} \mathbb{P}[X_{n+1} = u, X_n \in S_y] \\
&= \frac{1}{\mathbb{P}[X_n \in S_y]} \sum_{u \in S_z, v \in S_y} \mathbb{P}[X_{n+1} = u, X_n = v] \\
&= \frac{1}{\mathbb{P}[X_n \in S_y]} \sum_{u \in S_z, v \in S_y} p_{uv} \mathbb{P}[X_n = v] \\
&= q_{yz} \qquad (1)
\end{aligned}$$

where

$$q_{yz} = \begin{cases} \frac{1}{1+2\beta} & \text{if } z = y-1, y \neq 0 \\ \frac{2\beta}{1+2\beta} & \text{if } z = y+1, z \neq 0 \\ 0 & \text{otherwise.} \end{cases} \qquad (2)$$

Suppose for $y_j \in \mathbb{Z}_+$ with $j = 0, 1, \ldots, n$ we have $\mathbb{P}[Y_n = y_n, \ldots, Y_0 = y_0] > 0$, then for $y_{n+1} \in \mathbb{Z}_+$

$$\mathbb{P}[Y_{n+1} = y_{n+1} | Y_n = y_n, \ldots, Y_0 = y_0] = \mathbb{P}[X_{n+1} \in S_{y_{n+1}} | X_n \in S_{y_n}, \ldots, X_0 \in S_{y_0}]$$

$$= \sum_{x_{n+1} \in S_{y_{n+1}}} \mathbb{P}[X_{n+1} = x_{n+1} | X_n \in S_{y_n}, \ldots, X_0 \in S_{x_0}]$$

using the Markov Property for the Markov chain $\{X_n\}_{n \geq 1}$ we have that the above is

$$\begin{aligned}
\text{(why ?)} &= \mathbb{P}[X_{n+1} = S_{y_{n+1}} | X_n \in S_{y_n}] \\
&= \mathbb{P}[Y_{n+1} = y_{n+1} | Y_n = y_n] \qquad (3)
\end{aligned}$$

Using (3), (1), via an inductive argument and arguing similarly for case, $\mathbb{P}[Y_{n+1} = y_{n+1}, Y_n = y_n, \ldots, Y_0 = y_0] = 0$, we have

$$\mathbb{P}[Y_{n+1} = y_{n+1}, Y_n = y_n, \ldots, Y_0 = y_0] = \mathbb{P}(Y_0 = y_0) \prod_{i=1}^{n} q_{y_{i+1} y_i}.$$

Thus we have show that Y_n is a Markov chain on $\mathbb{Z}_+$ with transition matrix $Q = [q_{yz}]_{y,z \in \mathbb{Z}_+}$ and initial distribution given by the random variable Y_0.

Solution 1(b) (ii) : Let $\beta = \frac{1}{2}$. Using problem 1 in Homework 2, with $p_i = \frac{1}{2}$ we note that 1(c)(i) applies. So we have that $h^{\{0\}}(i) = \mathbb{P}_i(T_Y^{\{0\}} < \infty)) = 1$, where $T_Y^{\{0\}} = \min\{n \geq 0 : Y_n = 0\}$.

This means that the chain Y_n is recurrent.

Solution 1(b) (iii) : Let $\beta > \frac{1}{2}$. Using problem 1 in Homework 2, with $p_i = \frac{1}{1+2\beta}$, we note that 1(c)(ii) applies. So we have that $h^{\{0\}}(i) = \mathbb{P}_i(T_Y^{\{0\}} < \infty)) < 1$.

Thus, we get that Y_n is transient.

Solution 1(c) : We know, from part(a), that X_n is transient if $\beta < \frac{1}{2}$. As noted earlier, for $k \geq 0$, $X_n \in S_k$ if and only if $Y_n = k$. So,

$$\min\{n \geq 1 : X_n = \rho\} := R_X^{\{\rho\}} = R_Y^{\{0\}} := \min\{n \geq 1 : Y_n = 0\}.$$

So therefore,

$$\mathbb{P}_0(R_Y^{\{\rho\}} < \infty)) = 1 \text{ if and only if } \mathbb{P}_\rho(R_X^{\{\rho\}} < \infty)) = 1.$$

Therefore, (using Book keeping Exercise 1 in Homework 3), X_n is recurrent if and only if Y_n is recurrent. Thus from (b), X_n is recurrent if $\beta = \frac{1}{2}$ and transient if $\beta > \frac{1}{2}$.

2. Let $\xi, \xi_1, \xi_2, \ldots$ be i.i.d random variables, (denoting the number of people arriving in time unit i to a queue at a ticket counter) such that

$$P(\xi = k) = p_k, \qquad k = 0, 1, 2, \ldots,$$

with $\sum_{k=0}^{\infty} p_k = 1$. Let X_0 be the number of people in the queue at time 0. Then the number of people in the queue at time $n \geq 1$ can be described by

$$X_n = \max\{X_{n-1} - 1, 0\} + \xi_n.$$

(a) Verify that X_n is a Markov chain on state space $S = \{0, 1, 2, \ldots\}$.

(b) Show that the chain is irreducible if

$$0 < p_0 < 1 \text{ and if there exists } k > 1 \text{ such that } p_k > 0. \tag{4}$$

(c) Assume (4). Let $g : [0,1] \to [0,1]$ be given by $g(a) = \sum_{k=0}^{\infty} p_k a^k$. Let $\mu = g'(1)$.

i. Let $\mu \leq 1$ and define $f : S \to \mathbb{R}$ by $f(i) = i$. Show that

$$\mathbb{E}_i(f(X_1)) \leq f(i) \text{ for all } i \neq 0.$$

Conclude that X_n is recurrent.

ii. Show that if $\mu > 1$ then there is an $\beta \in (0, 1)$ such that $g(\beta) = \beta$

iii. Let $\mu > 1$ and define $f : S \to \mathbb{R}$ by $f(i) = \beta^i$. Show that

$$\mathbb{E}_i(f(X_1)) = f(i)$$

Conclude that X_n is transient.

Solution 2(a) : As $\mathbb{P}(\xi_n \geq 0) = 1$, by definition $\mathbb{P}(X_n \in S) = 1$, thus X_n takes values in S. For given $\{x_i\}_{1 \leq i \leq n}$, we note that the events

$$\{X_n = x_n, \ldots, X_0 = x_0\}$$

and

$$\{\xi_n = x_n - \max\{x_{n-1} - 1, 0\}, \ldots, \xi_1 = x_1 - \max\{x_0 - 1, 0\}, X_0 = x_0\}$$

are the same. Let $n \geq 1$ and assume that $\mathbb{P}(X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) > 0$.

$$\mathbb{P}[X_n = x_n | X_{n-1} = x_{n-1}, \ldots, X_0 = x_0] = \frac{\mathbb{P}[X_n = x_n, \ldots, X_0 = x_0]}{\mathbb{P}[X_{n-1} = x_{n-1}, \ldots, X_0 = x_0]}$$

$$= \frac{\mathbb{P}[\xi_n = x_n - \max\{x_{n-1} - 1, 0\}, \ldots, \xi_1 = x_1 - \max\{x_0 - 1, 0\}, X_0 = x_0]}{\mathbb{P}[\xi_{n-1} = x_{n-1} - \max\{x_{n-1} - 1, 0\}, \ldots, \xi_1 = x_1 - \max\{x_0 - 1, 0\}, X_0 = x_0]}$$

Using the Independence of $\{\xi_k\}_{k \geq 1}$, we have that the above is

$$= \frac{\prod_{i=1}^{n} \mathbb{P}[\xi_i = x_i - \max\{x_{i-1} - 1, 0\}] \cdot \mathbb{P}[X_0 = x_0]}{\prod_{i=1}^{n-1} \mathbb{P}[\xi_i = x_i - \max\{x_{i-1} - 1, 0\}] \cdot \mathbb{P}[X_0 = x_0]}$$

$$= \mathbb{P}[\xi_n = x_n - \max\{x_{n-1} - 1, 0\}].$$

$$= \frac{\mathbb{P}[X_n = x_n, X_{n-1} = x_{n-1}]}{\mathbb{P}[X_{n-1} = x_{n-1}]} \tag{5}$$

Further, as $\mathbb{P}[X_{n-1} = x_{n-1}] > 0$ we have

$$\begin{aligned} \mathbb{P}[X_n = x_n | X_{n-1} = x_{n-1}] &= \frac{\mathbb{P}[X_n = x_n, X_{n-1} = x_{n-1}]}{\mathbb{P}[X_{n-1} = x_{n-1}]} \\ &= \frac{\mathbb{P}[\xi_n = x_n - \max\{x_{n-1} - 1, 0\}, X_{n-1} = x_{n-1}]}{\mathbb{P}[X_{n-1} = x_{n-1}]} \\ &= \frac{\mathbb{P}[\xi_n = x_n - \max\{x_{n-1} - 1, 0\}]\mathbb{P}[X_{n-1} = x_{n-1}]}{\mathbb{P}[X_{n-1} = x_{n-1}]} \\ &= \mathbb{P}[\xi_n = x_n - \max\{x_{n-1} - 1, 0\}] \end{aligned} \tag{6}$$

From (5) and (6), we have that

$$\begin{aligned}
&\mathbb{P}[X_n = x_n | X_{n-1} = x_{n-1}, \dots, X_0 = x_0] = \mathbb{P}[X_n = x_n | X_{n-1} = x_{n-1}] \\
&= \mathbb{P}[\xi_n = x_n - \max\{x_{n-1} - 1, 0\}] \\
&= q_{x_n, x_{n-1}} := \begin{cases} p_{x_n} & \text{if } x_n \geq 0, x_{n-1} = 0 \\ p_{x_n - x_{n-1} + 1} & \text{if } x_n \geq x_{n-1} \geq 1 \\ p_0 & \text{if } x_n = x_{n-1} - 1, x_{n-1} \geq 1 \\ 0 & \\ \text{ otherwise} & \end{cases} \qquad (7)
\end{aligned}$$

Using (7), via an inductive argument and arguing similarly for case $\mathbb{P}[X_{n+1} = x_{n+1}, X_n = x_n, \dots, X_0 = x_0] = 0$ we have

$$\mathbb{P}[X_{n+1} = x_{n+1}, X_n = x_n, \dots, X_0 = x_0] = \mathbb{P}(X_0 = x_0) \prod_{i=1}^{n} q_{x_{i+1} x_i}$$

Thus we have show tnat X_n is a Markov chain on $\mathbb{Z}_+$ with transition matrix $Q = [q_{yz}]_{y,z \in \mathbb{Z}_+}$ given by

$$\begin{pmatrix} p_0 & p_1 & p_2 & p_3 & p_4 & p_5 & \dots \\ p_0 & p_1 & p_2 & p_3 & p_4 & p_5 & \dots \\ 0 & p_0 & p_1 & p_2 & p_3 & p_4 & \dots \\ 0 & 0 & p_0 & p_1 & p_2 & p_3 & \dots \\ 0 & 0 & 0 & p_0 & p_1 & p_2 & \dots \\ \vdots & & & \dots & & \dots & \dots \end{pmatrix}.$$

and initial distribution given by the random variable X_0.

□

Solution 2(b) : Let $0 < p_0 < 1$ and $K > 1$ be such that $p_K > 0$.
For $i \in \mathbb{Z}_+$ and let

$$S_i = \{k : \exists N \in \mathbb{N} \text{ such that } Q^N_{ik} > 0\}.$$

We will show that $S_i = \mathbb{Z}_+$. Firstly, note that $S_i \neq \emptyset$ as $q_{ij} = p_0 > 0$ if $j = i - 1$, implying $i - 1 \in S_i$.
Secondly, if $1 \leq j \in S_i$ then $j - 1 \in S_i$. This because, there exists an N such that $q^N_{ij} > 0$ and $q_{jk} = p_0 > 0$ for $k = j - 1 \in S_i$, which will imply that $q^{N+1}_{ik} > 0$.
Thirdly we will show that S_i is unbounded. If $j \in S$ then for $l = j + (K - 1)$ $q_{jl} = p_K > 0$ implying $j + (K - 1) \in S_i$. Repeating this inductively we have that $l + m(K - 1) \in S_i$ for all $m \geq 1$. With $K > 1$ this implies that S_i is unbounded.
Combining the three steps above we have that $S_i = \mathbb{Z}_+$ for all $i \geq 0$. Therefore the chain is irreducible.

□

Solution 2(c) : We shall need the following result stated in class.
Theorem 1 (Lyapunov Function) Let X_n be an irreducible Markov chain on $S = \{0\} \cup \mathbb{N}$, with transition matrix P. Then

(a) the chain is transient if and only if there is a bounded non-constant function $f : S \to \mathbb{R}$ such that

$$\sum_{j=0}^{\infty} p_{ij} f(j) = f(i), \text{ for all } i \neq 0 \qquad (8)$$

(b) the chain is recurrent if there is a function $f : S \to \mathbb{R}$ such that

$$\lim_{i \to \infty} f(i) = \infty \qquad (9)$$

and

$$\sum_{j=0}^{\infty} p_{ij} f(j) \le f(i), \;\text{ for all } i \neq 0 \tag{10}$$

We shall try to decide whether the chain is recurrent or transient by using Theorem 1. As indicated in the theorem, we need to find Lyapunov functions f satisfying (8) or (9) and (10).

Transience: Define $f : S \to \mathbb{R}$ by $f(i) = a^i$ for some real number $0 < a < 1$. We need to show $\sum_{j=0}^{\infty} p_{ij} f(j) = f(i)$ for $i \neq 0$, so we to find $0 < a < 1$ such that

$$\sum_{j=i-1}^{\infty} p_{j-i+1} a^j = a^i.$$

In other words, dividing both sides of above by a^{i-1}, we need to find a such that

$$a = \sum_{j=i-1}^{\infty} p_{j-i+1} a^{j-i+1},\; i \neq 0$$

or

$$a = \sum_{k=0}^{\infty} p_k a^k.$$

Solution 2(c) (i): Define $f : S \to \mathbb{R}$ by $f(i) = i$. Clearly f satisfies (9). Further,

$$\mathbb{E}_i(f(X_1) = \sum_{j=0}^{\infty} q_{ij} f(j) = \sum_{j=i-1}^{\infty} p_{j-i+1} j = \sum_{k=0}^{\infty} p_k (k + i - 1) = \mu + i - 1$$

The right hand side is less than equal to $f(i)$ as $\mu \le 1$. So f will satisfy (10) if $\mu \le 1$ and, via Theorem 1, the chain will be recurrent

Solution 2(c) (ii): Now $g(0) = p_0 > 0$, $g(1) = 1$ and $\mu = g'(1) = \sum_{k=0}^{\infty} k p_k > 1$. Note that it is a potentially divergent series, but all terms of the series are positive[1]

The function $g(x) - x$ is positive at $x = 0$, vanishes at $x = 1$ and is strictly increasing at $x = 1$ and so negative just to the left of $x = 1$.

Therefore, there must exist $0 < a_0 < 1$ such that $g(a_0) = a_0$.

□

Solution 2(c) (iii): Consider the function $f(i) = a_0^i$. Using (ii), for $i \neq 0$,

$$\mathbb{E}_i(f(X_1)) = \sum_{j=0}^{\infty} q_{ij} a_0^j = \sum_{j=i-1}^{\infty} p_{j-i+1} a^j = a_0^{i-1} g(a_0) = a_0^i = f(i).$$

Thus f, satisfies (8). So , via Theorem 1, establishing transience of the chain if $\mu > 1$.

□

In conclusion, the chain will be transient if $\mu > 1$ and recurrent otherwise.

[1]So $\mu > 1$ if it represents a real number and will assume condition if it diverges