Due: Friday March 5th, 2020

Problems to be turned in: 1,2

1. As discussed in class, let $\mathbb{T}_2$ be a rooted binary tree, with root ρ. For each $x \in \mathbb{T}_2$, let $\alpha(x) \in \mathbb{T}_2$ denote the ancestor of x and $\mid x \mid$ denote the distance to the root ρ. On $\mathbb{T}_2$, consider the weight function μ to be given by
$$\mu(\{x, \alpha(x)\}) = \beta^{|x|} \text{ for } x \in \mathbb{T}_2$$
where β is a positive number. Let X_n denote the canonical random walk on $(\mathbb{T}_2, \mu)$.
   (a) Show that for $\beta < \frac{1}{2}$ that the random walk X_n is recurrent.
   (b) Consider $Y_n = \mid X_n \mid$.
      i. Show that Y_n is a Markov Chain on $\mathbb{Z}_+$.
      ii. Show that Y_n is recurrent when $\beta = \frac{1}{2}$.
      iii. Show that Y_n is transient when $\beta > \frac{1}{2}$.
   (c) Conclude that X_n is transient if and only if $\beta > \frac{1}{2}$.

2. Let $\xi, \xi_1, \xi_2, \ldots$ be i.i.d random variables, (denoting the number of people arriving in time unit i to a queue at a ticket counter) such that
$$P(\xi = k) = p_k, \qquad k = 0, 1, 2, \ldots,$$
with $\sum_{k=0}^{\infty} p_k = 1$. Let X_0 be the number of people in the queue at time 0. Then the number of people in the queue at time $n \geq 1$ can be described by
$$X_n = \max\{X_{n-1} - 1, 0\} + \xi_n.$$
   (a) Verify that X_n is a Markov chain on state space $S = \{0, 1, 2, \ldots\}$.
   (b) Show that the chain is irreducible if
$$0 < p_0 < 1 \text{ and if there exists } k > 1 \text{ such that } p_k > 0. \tag{1}$$
   (c) Assume (1). Let $g : [0, 1] \to [0, 1]$ be given by $g(a) = \sum_{k=0}^{\infty} p_k a^k$. Let $\mu = g'(1)$.
      i. Let $\mu \leq 1$ and define $f : S \to \mathbb{R}$ by $f(i) = i$. Show that
$$\mathbb{E}_i(f(X_1)) \leq f(i) \text{ for all } i \neq 0.$$
      Conclude that X_n is recurrent.
      ii. Show that if $\mu > 1$ then there is an $\beta \in (0, 1)$ such that $g(\beta) = \beta$
      iii. Let $\mu > 1$ and define $f : S \to \mathbb{R}$ by $f(i) = \beta^i$. Show that
$$\mathbb{E}_i(f(X_1)) = f(i)$$
      Conclude that X_n is transient.

3. Consider the canopy tree $\mathcal{C}_2$ as a subgraph of $\mathbb{T}_2$ defined in class.
   (a) For $n \geq 1$, find the volume of the ball of radius n around ρ.
   (b) Show that there is only one infinite length self-avoiding path on $\mathcal{C}$.

4. Provide an example of a Markov Chain X_n on a graph $\Gamma = (V, E)$ such that chain has a stationary distribution π but π is not reversible.

Book Keeping Exercises

1. Let X_n be a random graph on weighted graph $\Gamma = (V, E)$ with weight function μ. For each $i \in V$, let $T_i = T^{\{i\}} = \min\{n \geq 0 : X_n = i\}$ and $R_i = \min\{n \geq 1 : X_n = i\}$ The following five conditions are equivalent:
   (a) There exists $i \in V$ such that $\mathbb{P}_i(R_i < \infty) = 1$.
   (b) For all $i \in V$, $\mathbb{P}_i(R_i < \infty) = 1$.
   (c) For all $i \in V$,
   $$\sum_{n=0}^{\infty} p_{ii}^n = \infty$$
   (d) For all $i, j \in V$, $\mathbb{P}_i(T_j < \infty) = 1$.
   (e) For all $i, j \in V$,
   $$\mathbb{P}_i(X \text{ hits } j \text{ infinitely often.}) = \mathbb{P}_i(\sum_{n=0}^{\infty} 1(X_n = j) = \infty) = 1.$$

2. Let X_n be a Markov chain on S with transition matrix P and initial distribution μ. Let $A \subset S$ and T^A be the hitting time of the set A, i.e $T^A = \min\{n \geq 0 : X_n \in A\}$.
   (a) For $n \geq 2$, show that
   $$P_i(T^A = n) = \sum_{i_1 \notin A} \sum_{i_2 \notin A} \cdots \sum_{i_{n-1} \notin A} \sum_{i_n \in A} p_{ii_1} \cdot \prod_{k=1}^{n-1} p_{i_k i_{k+1}}.$$
   and conclude that
   $$P_i(T^A = n) = \sum_{j \notin A} p_{ij} P_j(T^A = n-1).$$
   (b) Let $h^A : S \to [0, 1]$ be given by
   $$h^A(i) = P_i(T^A < \infty).$$
   Using 2(a) and the fact $\{T^A < \infty\} = \bigcup_{n=1}^{\infty} \{T^A = n\}$, show that h^A is a solution of the linear system of equations given by
   $$h^A(i) = \begin{cases} 1 & \text{if } i \in A \\ \sum_{j \in S} p_{ij} h^A(j) & \text{if } i \notin A \end{cases} \tag{2}$$
   (c) Let $f : S \to [0, \infty)$ be another solution to (2).
      i. Show that for each $i \in A^c$,
      $$f(i) = P_i(T^A = 1) + P_i(T^A = 2) + \sum_{j \notin A} \sum_{k \notin A} p_{ij} p_{jk} f(k)$$
      and hence conclude that
      $$f(i) = \sum_{m=1}^{n} P_i(T^A = m) + \sum_{i_m \in A^c, 1 \leq m \leq n} p_{ii_1} \prod_{m=2}^{n} p_{i_{m-1} i_m} f(i_n).$$
      ii. Show that $f(i) \geq h^A(i)$ for all $i \in V$