1. Let X_n be a Markov chain on $S = \{0, 1, \dots\}$ with transition matrix P with

$$p_{ij} = \begin{cases} 1 & \text{if } i = j = 0 \\ p_i & \text{if } i \geq 1, j = i+1 \\ 1 - p_i & \text{if } i \geq 1, j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

where we assme $0 < p_i < 1$ for every $i \geq 1$. Let $A = \{0\}$ and let $h^A : S \to [0,1]$ to be given by

$$h^A(i) = P_i(T^A < \infty).$$

(a) Let $u_i = h^A(i-1) - h^A(i)$. Show that

$$u_{i+1} = \frac{q_i}{p_i} u_i,$$

for $i \geq 1$.

(b) Conclude that for $i \geq 1$,

$$h^A(i) = 1 - u_1 \left[\sum_{j=1}^{i} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right) \right].$$

(c) Further,

i. If $\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right) = \infty$, then

$$h^A(i) = 1 \text{ for all } i \geq 0.$$

ii. If $\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right) < \infty$, then

$$h^A(i) = \frac{\sum_{j=i+1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right)}{\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right)} \qquad \text{for all } i \geq 0.$$

Solution 1. We will use the following proposition.

Proposition 1: *Let X_n be a Markov chain on S with transition matrix P and initial distribution μ. Let $A \subset S$ and T^A be the hitting time of the set A. Let $h^A : S \to [0,1]$ be given by*

$$h^A(i) = P_i(T^A < \infty).$$

(A) h^A is a solution of the linear system of equations given by

$$h^A(i) = \begin{cases} 1 & \text{if } i \in A \\ \sum_{j \in S} p_{ij} h^A(j) & \text{if } i \notin A \end{cases} \tag{1}$$

(B) If $f : S \to [0, \infty)$ is another solution to (1) then $f(i) \geq h(i)$ for all $i \in S$.

Solution 1(a): We shall denote $q_i = 1 - p_i$. Note Using the above Proposition 1 (A), we know that for $i > 0$:

$$h^A(i) = h^A(i+1)p_i + h^A(i-1)q_i$$

This is the same as

$$p_i h^A(i) + q_i h^A(i) = p_i h^A(i) + q_i h^A(i-1).$$

that $q_i, p_i > 0$ for all $i > 0$. Let $u_i = h^A(i-1) - h^A(i)$ and a simple rearrangement implies that

$$u_{i+1} = \frac{q_i}{p_i} u_i, \text{ for } i \geq 1. \tag{2}$$

□

Solution 1(b): Iterating (2) inductively we obtain for all $i \geq 1$

$$u_{i+1} = \left(\prod_{k=1}^{i} \frac{q_k}{p_k} \right) u_1$$

On the other hand it is easy to see via a telescoping sum argument that

$$h^A(0) - h^A(i) = \sum_{j=1}^{i} u_j,$$

and this implies

$$h^A(i) = 1 - u_1 \left[\sum_{j=1}^{i} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right) \right], \qquad \text{for all } i \geq 1. \tag{3}$$

□

Solution 1(c)(i): We assume that $\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right) = \infty$.
If $u_1 \neq 0$, by assumption there exists an $N > 1$ such that

$$\sum_{j=1}^{N} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right) > \frac{1}{\mid u_1 \mid}.$$

Substituting this in (3) with $i = N$, would contradict the fact that $0 \leq h^A(N) \leq 1$. Hence, $u_1 = 0$. So from (3) again we have that

$$h^A(i) = 1 \text{ for all } i \geq 1.$$

By definition $h^A(0) = 1$ and we are done.

□

Solution 1(c)(ii): $\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right) < \infty$.

Using the assumption and the fact that $0 \leq h^A(i) \leq 1$ in (3), we have that

$$0 \leq u_1 \leq \frac{1}{\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right)}.$$

By Proposition 1 (B), h is the minimal solution. Using this and the above in (3) this implies that

$$u_1 = \frac{1}{\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right)}.$$

Therefore for $i \geq 1$

$$h^A(i) = \frac{\sum_{j=i+1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k}\right)}{\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k}\right)}.$$

By definition $h^A(0) = 1$ and we are done. □

5. Let S be a countable set, $\mathcal{F} = \mathcal{P}(S)$ be the power set of S, $\mathbb{P}$,$\mathbb{Q}$, $\mathbb{R}$ be Probabilities on S.
   (a) Let $G = \{s \in S : \mathbb{P}(\{s\}) \geq \mathbb{Q}(\{s\})\}$, show that for any $F \in \mathcal{F}$ we have
      i. $\mathbb{P}(F) - \mathbb{Q}(F) \leq \mathbb{P}(F \cap G) - \mathbb{Q}(F \cap G) \leq \mathbb{P}(G) - \mathbb{Q}(G)$
      ii. $\mathbb{Q}(F) - \mathbb{P}(F) \leq \mathbb{Q}(G^c) - \mathbb{P}(G^c)$
      iii. $\mathbb{P}(G) - \mathbb{Q}(G) = \mathbb{Q}(G^c) - \mathbb{P}(G^c)$
   (b) Conclude from part (a) that

$$\| \mathbb{P} - \mathbb{Q} \|_{\text{TV}} := \max_{A \in \mathcal{F}} | \mathbb{P}(A) - \mathbb{Q}(A) | = \frac{1}{2} \sum_{s \in S} | \mathbb{P}(\{s\}) - \mathbb{Q}(\{s\}) | .$$

Solution 5: We make some observations before we begin. Note that for $s \notin G$ we have

$$\mathbb{P}(\{s\}) < \mathbb{Q}(\{s\}).$$

In particular, for $S \subset G$ we have $\mathbb{P}(S) \geq \mathbb{Q}(S)$ and for $S \subset G^c$ we have $\mathbb{P}(S) < \mathbb{Q}(S)$. Let us make a notation and write $(\mathbb{P} - \mathbb{Q})(F) = \mathbb{P}(F) - \mathbb{Q}(F)$ for any F. With this, for $F \subset G$ we have $(\mathbb{P} - \mathbb{Q})(F) \geq 0$ and for $F \subset G^c$ we have $(\mathbb{P} - \mathbb{Q})(F) < 0$.

Solution 5(a) (i) Let $F \in \mathcal{F}$, and write $F = (F \cap G) \cup (F \cap G^c)$. By disjointness,

$$(\mathbb{P} - \mathbb{Q})(F) = (\mathbb{P} - \mathbb{Q})(F \cap G) + (\mathbb{P} - \mathbb{Q})(F \cap G^c)$$

Note that $F \cap G^c \subset G^c$ so by our observation the second term is negative hence

$$(\mathbb{P} - \mathbb{Q})(F) \leq (\mathbb{P} - \mathbb{Q})(F \cap G).$$

In similar fashion, note that

$$(\mathbb{P} - \mathbb{Q})(G) = (\mathbb{P} - \mathbb{Q})(G \cap F) + (\mathbb{P} - \mathbb{Q})(G \cap F^c)$$

As $G \cap F^c \subset G$ the latter term on the RHS is non-negative hence

$$(\mathbb{P} - \mathbb{Q})(G) \geq (\mathbb{P} - \mathbb{Q})(G \cap F).$$

□

Solution 5(a) (ii) Now,

$$(\mathbb{P} - \mathbb{Q})(G^c) = (\mathbb{P} - \mathbb{Q})(G^c \cap F) + (\mathbb{P} - \mathbb{Q})(G^c \cap F^c)$$

and the latter term is negative as the argument is a subset of G^c hence

$$(\mathbb{P} - \mathbb{Q})(G^c) \leq (\mathbb{P} - \mathbb{Q})(G^c \cap F).$$

However, see that

$$(\mathbb{P} - \mathbb{Q})(F) = (\mathbb{P} - \mathbb{Q})(G^c \cap F) + (\mathbb{P} - \mathbb{Q})(G \cap F)$$

and the latter term is non-negative as the argument is a subset of G, hence

$$(\mathbb{P} - \mathbb{Q})(F) \geq (\mathbb{P} - \mathbb{Q})(G^c \cap F).$$

Combining the two inequalities we are done.

□

Solution 5(a) (iii) We have

$$0 = (\mathbb{P} - \mathbb{Q})(S) = (\mathbb{P} - \mathbb{Q})(G) + (\mathbb{P} - \mathbb{Q})(G^c)$$

which implies that $\mathbb{P}(G) - \mathbb{Q}(G) = \mathbb{Q}(G^c) - \mathbb{P}(G^c)$.

□

Solution 5(b): Note that

$$\sum_{s \in S} |(\mathbb{P}-\mathbb{Q})(\{s\})| = \sum_{s \in G} |(\mathbb{P}-\mathbb{Q})(\{s\})| + \sum_{s \in G^c} |(\mathbb{P}-\mathbb{Q})(\{s\})| = (\mathbb{P}-\mathbb{Q})(G) - (\mathbb{P}-\mathbb{Q})(G^c) = 2(\mathbb{P}-\mathbb{Q})(G) \tag{4}$$

Now, from **5 (a)(i)**, for any $A \subset \mathcal{F}$ we already know that

$$(\mathbb{P} - \mathbb{Q})(A) \leq (\mathbb{P} - \mathbb{Q})(G). \tag{5}$$

Using (4) and (5) we have that

$$\| \mathbb{P} - \mathbb{Q} \|_{\text{TV}} := \max_{A \in \mathcal{F}} | \mathbb{P}(A) - \mathbb{Q}(A) | = (\mathbb{P} - \mathbb{Q})(G) = \frac{1}{2} \sum_{s \in S} |(\mathbb{P} - \mathbb{Q})(\{s\})|.$$

□