Due: Friday February 26th, 2020
Problems to be turned in: 2,5

1. Let $0 < \alpha, \beta < 1$. Consider a Markov chain X_n on state space $S = \{0, 1\}$,with

$$\mathbb{P}(X_0 = 0) = \mu_0 \in (0,1), \mathbb{P}(X_0 = 1) = \mu_1 \in (0,1),$$

and transition matrix

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}.$$

   (a) For any $n \geq 0$, show that

$$\mathbb{P}(X_n = 1) = \frac{\alpha}{\alpha+\beta} + (1-\alpha-\beta)^n \left(\mu_1 - \frac{\alpha}{\alpha+\beta}\right)$$

   (b) Find the stationary distribution π of X_n.

   (c) Let $1, \lambda$ be the eigen values of P. Show that

$$\| \mathbb{P} \circ X_n^{-1} - \pi \|_{\text{TV}} = \lambda^n \left(\mu_1 - \frac{\alpha}{\alpha+\beta}\right)$$

2. Let X_n be a Markov chain on $S = \{0, 1, \dots\}$ with transition matrix P with

$$p_{ij} = \begin{cases} 1 & \text{if } i = j = 0 \\ p_i & \text{if } i \geq 1, j = i+1 \\ 1 - p_i & \text{if } i \geq 1, j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

where we assme $0 < p_i < 1$ for every $i \geq 1$. Let $A = \{0\}$ and let $h^A : S \to [0,1]$ to be given by

$$h^A(i) = P_i(T^A < \infty).$$

   (a) Let $u_i = h^A(i-1) - h^A(i)$. Show that

$$u_{i+1} = \frac{q_i}{p_i} u_i,$$

   for $i \geq 1$.

   (b) Conclude that for $i \geq 1$,

$$h^A(i) = 1 - u_1 \left[\sum_{j=1}^{i} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k}\right)\right].$$

   (c) Further,

      i. If $\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k}\right) = \infty$, then

$$h^A(i) = 1 \text{ for all } i \geq 0.$$

ii. If $\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k}\right) < \infty$, then

$$h^A(i) = \frac{\sum_{j=i+1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k}\right)}{\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k}\right)}.$$

3. Let $a, b, c \in \mathbb{R}$ and $D = b^2 - 4ac$. Let $\{x_n\}_{n \geq 1}$ be a sequence such that for $n \geq 2$ it satisfies:

$$ax_n + bx_{n-1} + cx_{n-2} = 0. \tag{1}$$

   (a) Let $D > 0$ and α, β be the two distinct roots of the quadratic equation $ax^2 + bx + c$. Show that

   $$x_n = A\alpha^n + B\beta^n,$$

   where $A, B \in \mathbb{R}$ is a solution to (1) and any solution to (1) is of this form.

   (b) Let $D = 0$ and α be the real repeated root of the quadratic equation $ax^2 + bx + c$. Show that

   $$x_n = (A + nB)\alpha^n$$

   is a solution to (1) and any solution to (1) is of this form.

   {*Hint for the second part of (a) and (b): if y_n is any other solution, solve for A, B by setting $x_0 = y_0$ and $x_1 = y_1$ then use the fact that both x_n and y_n are solutions to conclude $a(x_n - y_n) + b(x_{n-1} - y_{n-1}) + c(x_{n-2} - y_{n-2}) = 0$. Argue inductively that $x_n = y_n$ for all $n \geq 2$.*}

4. Let X_n be a Markov chain on state space $S = \{0, 1, \ldots\}$ with $\mathbb{P}(X_0 = 10) = 1$ and transition matrix P, given by

$$p_{ij} = \begin{cases} 1 & \text{if } i = j = 0 \\ p & \text{if } i \geq 1, j = i + 1 \\ 1 - p & \text{if } i \geq 1, j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

   Let $A = \{0\}$. Using Exercise 3 to find $\mathbb{P}(T^A < \infty)$.

5. Let S be a countable set, $\mathcal{F} = \mathcal{P}(S)$ be the power set of S, $\mathbb{P}$,$\mathbb{Q}$, $\mathbb{R}$ be Probabilities on S.

   (a) Let $G = \{s \in S : \mathbb{P}(\{s\}) \geq \mathbb{Q}(\{s\})\}$, show that for any $F \in \mathcal{F}$ we have

      i. $\mathbb{P}(F) - \mathbb{Q}(F) \leq \mathbb{P}(F \cap G) - \mathbb{Q}(F \cap G) \leq \mathbb{P}(G) - \mathbb{Q}(G)$

      ii. $\mathbb{Q}(F) - \mathbb{P}(F) \leq \mathbb{Q}(G^c) - \mathbb{P}(G^c)$

      iii. $\mathbb{P}(G) - \mathbb{Q}(G) = \mathbb{Q}(G^c) - \mathbb{P}(G^c)$

   (b) Conclude from part (a) that

   $$\| \mathbb{P} - \mathbb{Q} \|_{\text{TV}} := \max_{A \in \mathcal{F}} | \mathbb{P}(A) - \mathbb{Q}(A) | = \frac{1}{2} \sum_{s \in S} | \mathbb{P}(\{s\}) - \mathbb{Q}(\{s\}) | .$$

Book Keeping Exercises

Let μ be any probability[1] on S. Let $P = [p_{ij}]_{i,j \in S}$ be a matrix for which the entries satisfiy $p_{ij} \in [0,1]$ for all $i, j \in S$ and $\sum_{k=1}^{\infty} p_{ik} = 1$ for each $i \in S$. Let $(\Omega, \mathcal{F}, P)$ be a probability space and for $n \geq 0$, let $X_n : \Omega \to S$ be a sequence of random variables whose joint distribution is given by

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_k = i_k) = \mu(\{i_0\}) p_{i_0 i_1} p_{i_1 i_2} \ldots p_{i_{k-1} i_k}, \tag{2}$$

for all $k \geq 0$. Then X_n is said to be a Markov chain on state space S, with initial distribution μ, and transition matrix P.

1. (Distribution of X_n) Using the definition of Probability, (2), and properties of the transition matrix P show that

$$P(X_n = j) \quad = \sum_{i_0, i_1, \ldots i_{n-1} \in S} \mu(\{i_0\}) \left(\prod_{k=1}^{n-1} p_{i_{k-1} i_k} \right) p_{i_{n-1}, j}. \tag{3}$$

   In particular note that the distribution of X_n is determined soley by μ and the transition matrix P.

2. (Markov Property) Every Markov chain has an underlying memoryless property. For $n \in \mathbb{N}$ let $i_0, i_1, \ldots, i_{n-2}, i, j \in S$ is such that

$$P(X_{n-1} = i, X_{n-2} = i_{n-2}, \ldots, X_0 = i_0) > 0,$$

   then

$$P(X_n = j | X_{n-1} = i, X_{n-2} = i_{n-2}, \ldots, X_0 = i_0) = P(X_n = j | X_{n-1} = i)$$

   Further, for all $m \geq 1, n \geq 0$, suppose Y_n is a sequence of random variables such that

$$Y_n \sim (X_{m+n} \mid X_m = i)$$

   Then Y_n is a Markov chain on state space S with transition matrix P and $P(Y_0 = i) = 1$. Further the sequence $\{Y_n\}_{n \geq 0}$ is independent of the random variables $\{X_0, X_1, X_2, \ldots, X_{m-1}\}$.

3. Let P^n be the n-th power of the matrix P and $p_{ij}^{[n]}$ denote the (i,j)-th element of P^n. Show that

$$P(X_n = j \mid X_0 = i) = p_{ij}^{[n]}$$

   and

$$P(X_n = j | X_{n-1} = i) = p_{ij}$$

   if $P(X_{n-1} = i) > 0$.

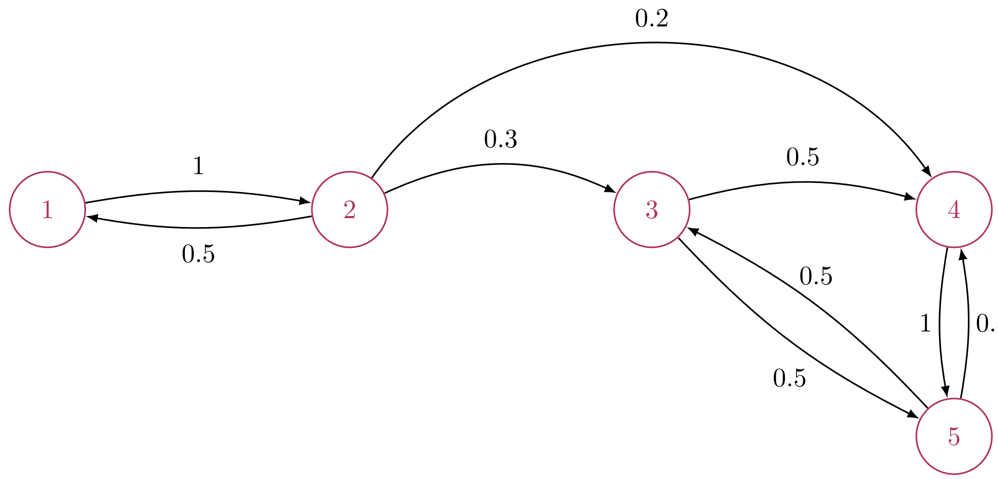
4. (Chapman-Kolmogrov Equation) For $r, s \geq 0$ and $r+s = n$, we have $P^n = P^{r+s} = P^r P^s$. Conclude

$$P(X_n = j | X_0 = i) = \sum_{k \in S} P(X_r = k | X_0 = i) P(X_s = j | X_0 = k)$$

   where the sum is understood to include only those terms for which the conditional probabilities are defined. This is referred to as the Chapman-Kolmogorov equation.

5. Consider a Markov chain X_n on state space $S = \{1, 2, 3, 4, 5\}$, $X_0 \sim$ Uniform $\{1, 2, 3, 4, 5\}$, and the dynamics of the chain be given via a directed graph with S as the vertex set.

[1] i.e $\mu : \mathcal{P}(S) \to [0,1]$ such that μ satisfies Definition **??**, where $\mathcal{P}(S)$ is the power set of S.



Suppose P is the transition matrix of the above chain. Show that

$$\lim_{n\to\infty} p_{11}^{[n]} = 0, \qquad \lim_{n\to\infty} p_{12}^{[n]} = 0, \qquad \lim_{n\to\infty} p_{13}^{[n]} = \frac{2}{7}, \qquad \lim_{n\to\infty} p_{14}^{[n]} = \frac{1}{7}, \qquad \lim_{n\to\infty} p_{15}^{[n]} = \frac{4}{7}.$$