Due: Thursday April 29th, 2021, 10pm
Problems to be turned in: 3,4

1. Consider a Galton-Watson process with immigration, that is
$$Z_n = 1(Z_{n-1} \neq 0) \sum_{i=1}^{Z_{n-1}} X_i^n + I_n,$$
where $\{X_i^n\}_{1 \leq i,n}$ are i.i.d X such that $X \sim$ Bernoulli(p) and $\{I_n\}_{n \geq 0}$ are i.i.d. Poisson (λ). Find a_n, b_n such that $M_n = a_n(Z_n - b_n)$ is a martingale.

2. Let S_n be the simple symmetric walk on $\mathbb{Z}^d$. Let
$$\tau_R = \inf\{n \geq 0 : \mid S_n \mid = R\}.$$
Let $h : \mathbb{Z}^d \to [0, \infty)$ be given by
$$h(x) = \mathbb{P}_x(\tau_{20} < \tau_1).$$
Show that
   (a) $h(x) = 1$ whenever $\mid x \mid \geq 20$
   (b) $h(x) = 0$ whenever $\mid x \mid \leq 1$
   (c) h is harmonic on the set $1 < \mid x \mid < 20$, i.e.
$$h(x) = \frac{1}{2d} \left(\sum_{i=1}^{d} h(x + e_d) + h(x - e_d) \right),$$
   whenever $1 < \mid x \mid < 20$, where $\{e_i : 1 \leq i \leq d\}$ are the standard basis for $\mathbb{Z}^d$.

3. Consider the graph Γ be two copies of $\mathbb{Z}^3$ joined at the origin. That is Γ is the graph obtained by taking copies $\mathbb{Z}^3_{(1)}$ and $\mathbb{Z}^3_{(2)}$ of $\mathbb{Z}^3$ with origins denoted by O_1 and O_2 being joined by an edge. Let S_n be the simple symmetric walk on Γ with all edges having weight 1. Let $h : \Gamma \to [0, \infty)$ be given by
$$h(x) = \mathbb{P}_x(\cup_{n=1}^{\infty} \cap_{m=n}^{\infty} \{S_m \in \mathbb{Z}^3_{(1)}\}).$$
Show that
   (a) h is non-negative harmonic on Γ.
   (b) h does not satisfy the Liouville Property.

4. In the Moran model we may introduce a selective bias by making it twice as likely that a type a individual is chosen to die, as compared to a type A individual. Thus in a population of size m containing i type A individuals, the probability that some type A is chosen to die is now $\frac{i}{i+2(m-i)}$. Suppose we begin with just one type A. What is the probability that eventually the whole population is of type A?

Book Keeping Exercises

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let

$$L_2 = \{X : \Omega \to \mathbb{R} \,|\, \mathbb{E}[X^2] < \infty\}.$$

Suppose $\{\xi_n\}_{n\geq 1}$ and ξ are in L_2 such that $E[(\xi_n - \xi)^2] \to 0$ as $n \to \infty$. Show that $E(\xi_n) \to E(\xi)$ as $n \to \infty$.

2. Let $\mathcal{F} = \{F : \mathbb{R} \to [0,1] : F$ is a distribution function.$\}$ Define the function $d : \mathcal{F} \times \mathcal{F} \to [0, \infty)$ by

$$d(F,G) = \inf\{\epsilon > 0 : G(x-\epsilon) - \epsilon \leq F(x) \leq G(x+\epsilon) + \epsilon\}.$$

Show that $(\mathcal{F}, d)$ is a metric space. Further show that a sequence of random variables $\{X_n\}$ converges in distribution to X if and only if $\rho(F_{X_n}, F_X) \to 0$ as $n \to \infty$.

3. Let $\mathcal{X}$ be the set of all random variables on the probability space $(\Omega, \mathcal{B}, \mathbb{P})$. Define a function $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\rho(X,Y) = E(\min(|X-Y|, 1),$$

for any $X, Y \in \mathcal{X}$. Show that $(\mathcal{X}, \rho)$ is a metric space. Further show that a sequence of random variables $\{X_n\}$ converges in probability to X if and only if $\rho(X_n, X) \to 0$ as $n \to \infty$.

4. Let $\mathcal{X}$ be the set of all random variables on the probability space $(\Omega, \mathcal{B}, \mathbb{P})$ such that $\mathbb{E}[X^2] < \infty$. Define a function $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\eta(X,Y) = \sqrt{E(|X-Y|^2)}$$

for any $X, Y \in \mathcal{X}$.

   (a) Show that $(\mathcal{X}, \eta)$ is a metric space.

   (b) Show that a sequence of random variables $\{X_n\}$ converges to X in $(\mathcal{X}, \eta)$, i.e. $\eta(X_n, X) \to 0$ then $\rho(X_n X) \to 0$ as $n \to \infty$.