Due: Thursday April 22n, 2021, 10pm
Problems to be turned in: 3,4,7

1. Let $\{Y_k\}_{k\geq 1}$ be a sequence of discrete random variables. Let $\mathcal{A}_k$ be the filtration of observable events by time k w.r.t $\{Y_k\}_{k\geq 1}$. Let X be another random variable such that $E[\mid X \mid] < \infty$. Define $X_n = E[X \mid \mathcal{A}_n]$
   (a) Show that $\{X_n\}_{n\geq 1}$ is a martingale w.r.t. the filtration $\mathcal{A}_n$
   (b) Does X_n converge as $n \to \infty$?
2. (Revisit Q2 Hw 6) Let $p = \frac{1}{2}$ and for $n \geq 1$ let
$$Y_n = \frac{1}{2} + \sum_{k=1}^{n} \frac{1}{2^{k+1}} X_k.$$
   Show that $Y_n \to Y$ converges as $n \to \infty$. Can you characterise the distribution Y ?
3. Let Ω be a finite set. For $m \geq 1$, let $\psi \in \Omega^m$. For $n \geq 1$. let $\{X_n\}_{n\geq 1}$ be independent and identically distributed as Uniform(Ω). Consider
$$T = \min\{k \geq m : (X_{k-m+1}, X_{k-m+2}, \dots, X_k) = \psi\}.$$
   Show that there exists a $C > 0$ and a Geometric random variable S such that $T \leq CS$.
4. Let X_n be a Markov chain on S with transition matrix P and initial distribution μ. Let $\mid S \mid < \infty$ and $\{X_n\}$ be irreducible and aperiodic. For $n \geq 1$, let
$$\bar{d}(n) = \max_{x,y\in S} \parallel \mathbb{P}_x(X_n \in \cdot) - \mathbb{P}_y(X_n \in \cdot) \parallel_{\mathrm{TV}}.$$
   Show that $\bar{d}(n+m) \leq \bar{d}(n)\bar{d}(m)$.
5. Let $\{M_n\}_{n\geq 1}$ be a martingale and T be stopping time for $\{M_n\}_{n\geq 1}$. Suppose $\mathbb{E}(T) < \infty$ and thre exists $C > 0$ such that for all $n \geq 1$,
$$\mid M_n - M_{n-1} \mid \leq C$$
   then show that
$$\mathbb{E}(M_T) < \infty \text{ and } \lim_{n\to\infty} \mathbb{E}(M_n \mid T \geq n)\mathbb{P}(T \geq n) = 0.$$
6. Let $\{X_n\}_{n\geq 1}$ be independent with
$$X_n = \begin{cases} 1 & \text{with probability } \frac{1}{2n} \\ 0 & \text{with probability } 1 - \frac{1}{n} \\ -1 & \text{with probability } \frac{1}{2n} \end{cases}$$
   For $n \geq 1$,
$$Y_n = \begin{cases} X_n & \text{if } n = 1 \text{ or } Y_{n-1} = 0, n \geq 2 \\ nY_{n-1} \mid X_{n-1} \mid & \text{if } Y_{n-1} \neq 0, n \geq 2. \end{cases}$$
   (a) Show that Y_n is a martingale.
   (b) Find $\lim_{n\to\infty} \mathbb{P}(Y_n = 0)$.

(c) Decide if Martingale convergence theorem applies and whether Y_n converges to some Y as $n \to \infty$ with probability 1.

7. Let $m \in \mathbb{N}$ and $m \geq 2$. At time $n = 1$, an urn contains $2m$ balls, of which m are red and m are blue. At each time $n = 2, 3, \ldots m$ we draw a randomly chosen ball from the urn and record its colour. We do not replace it. Therefore, at time n the urn contains $2m - n + 1$ balls. For $n = 1, \ldots 2m$ let N_n denote the number of red balls and f_n denote the fraction of red balls in the urn after time n.

   (a) Show that f_n is a martingale.

   (b) Let T be the first time at which the ball drawn is red. Find $\mathbb{E}(f_T)$.