Mixing Time of Random Walk

Markov chain — i) defined on a finite space

ii) irreducible

Fundamental Result: i + ii) ensures that the Markov chain has a unique stationary distn. π

$\oplus$ aperiodic, the t-th time distribution of the Markov chain converges to π as $t \to \infty$

does not depend on the initial distribution

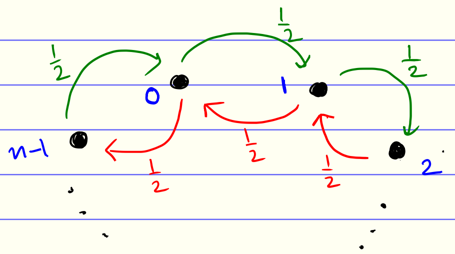
How fast does it converge?

Setup:

simple random walk on the n-cycle

$\mathbb{Z}_n$ = the set of integers modulo n = $\{0, 1, 2, \dots, n-1\}$.

$$P(i,j) = \begin{cases} \frac{1}{2} & j \equiv i+1 \bmod n \\ \frac{1}{2} & j \equiv i-1 \bmod n \\ 0 & \text{otherwise} \end{cases}$$



$t \in \mathbb{N} \cup \{0\}$, $\{Z_t\}_{t \geq 0}$ be the Markov chain on $\mathbb{Z}_n$ with transition matrix P

i) defined on finite space $\mathbb{Z}_n$

ii) irreducible

We have unique stationary distn $\pi = \left(\frac{1}{n} : i \in \mathbb{Z}_n\right)$

Total Variation Distance: For any two probability distributions μ and ν on space $\mathcal{X}$.

$$\|\mu - \nu\|_{TV} = \max_{A \subseteq \mathcal{X}} |\mu(A) - \nu(A)|$$

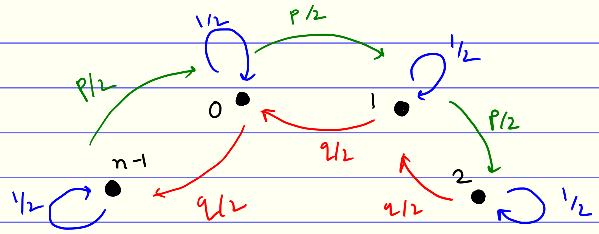
Proposition: (Convergence) Suppose that a finite space ($\mathcal{X}$) Markov chain $\{X_t\}_{t\geq 0}$ with transition probability matrix P is irreducible and aperiodic with stationary distn π.

Then $\exists\ \alpha \in (0,1)$ and $C > 0$

$$\text{s.t.} \quad \underbrace{\max_{x \in \mathcal{X}} \| P(X_t \in \cdot \mid X_0 = x) - \pi(\cdot) \|_{TV}}_{d(t)} \leq C\alpha^t$$

Choose $0 < p < 1$, $q = 1-p$.

$$\tilde{P} = \begin{cases} \frac{p}{2} & j \equiv i+1 \mod n \\ \frac{q}{2} & j \equiv i-1 \mod n \\ \frac{1}{2} & j \equiv i \\ 0 & \text{otherwise} \end{cases}$$



$\{Z_t\}_{t\geq 0}$ with $\tilde{P}$ is $\longrightarrow$ i) finite space Markov chain
ii) irreducible
iii) aperiodic

$\longrightarrow$ Lazy $(p-q)$ biased random walk on n-cycle

From now on we will denote $\tilde{P}$ by P.

$\pi(i) = \left(\frac{1}{n} : i \in \mathbb{Z}_n\right)$ works here.

$\exists\ C > 0$ and $\alpha \in (0,1)$ s.t. $\max_{i \in \mathbb{Z}_n} \| P(Z_t \in \cdot \mid Z_0 = i) - \pi \|_{TV} \leq C\alpha^t$

has dependence on $|\mathbb{Z}_n| = n$

"long time" to be within small enough distance to stationarity

$\downarrow$

in terms of $|\mathbb{Z}_n| = n$

Idea: 1) fix a target distance to reach within stationarity.

2) find out how the time or steps taken to reach this target distance changes with n

$$d(t) := \max_{i \in \mathbb{Z}_n} \| P(Z_t \in \cdot \mid Z_0 = i) - \pi(\cdot) \|_{TV}$$

i) $d(t) \in [0,1]$

ii) non-increasing in t

iii) $d(t+s) \leq 2\, d(t)\, d(s)$

later (given in notes)

Mixing time.

$$t_{mix} = \min\{t : d(t) \le 1/4\}$$

→ well accepted standard. $< \frac{1}{2}$ also works.

$t \ge t_{mix}$, we accept that the Markov chain is close enough to stationarity.

Theorem 1: For the above setup of $\{Z_t\}_{t\ge 0}$,

$$\frac{n^2}{32} \le t_{mix} \le n^2$$

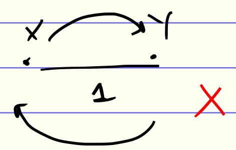
Overview of the proof of Thm 1:

Upper bound: We consider two particles X and Y performing the lazy $(p-q)$ biased random walk on n-cycle

The particle X starts from x

" " Y starts from y.

Avoid a situation where X and Y are unit distance apart and they jump over one another



Make sure that the two particles never move simultaneously.

How the particles move:

① Until the particles meet/collide,

Toss a fair coin
- $\frac{1}{2}$ heads → particle X
  - moves clockwise wp p
  - moves anticlockwise wp q.
- $\frac{1}{2}$ tails → particle Y
  - moves clockwise wp p
  - moves anticlockwise wp q.

② When they meet,

The particle X and Y moves together
- clockwise $\frac{p}{2}$
- anticlockwise $\frac{q}{2}$
- stays in the same position $\frac{1}{2}$

See that

$$\text{the particle } X \begin{cases} \text{move clockwise} & p/2 \\ \text{moves anticlockwise} & q/2 \\ \text{stay} & \frac{1}{2} \end{cases} \quad \text{— the particle } Y$$

both X and Y marginally perform the lazy $(p-q)$ biased random walk on n-cycle

This is called a coupling of the lazy $(p-q)$ biased random walk on n-cycle

Note: After they have met, they stay together

Let X_t, Y_t denote the position of X and Y at time t

Then $\{X_t\}_{t\geq 0}$, $\{Y_t\}_{t\geq 0}$ are Markov chains on $\mathbb{Z}_n$ with P

$\{(X_t, Y_t)\}_{t\geq 0}$ is also a Markov chain on $\mathbb{Z}_n \times \mathbb{Z}_n$

$\{(X_t, Y_t)\}_{t\geq 0}$ is a coupling of the transition matrix P.

Now we define the coupling time

$$\tau_{couple} = \min\{t : X_s = Y_s \quad \forall\, s \geq t\}$$

$$= \text{first time when } X \text{ and } Y \text{ meet.}$$

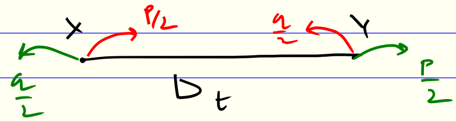
Propn 1: $$d(t) \leq \max_{x,y\in\mathbb{Z}_n} \mathbb{P}_{x,y}(\tau_{couple} > t)$$

$$t_{mix} \leq 4 \max_{x,y\in\mathbb{Z}_n} \mathbb{E}_{x,y}(\tau_{couple})$$

Define $D_0 = \min\{a \in \mathbb{N}\cup\{0\} \mid y \equiv x + a \mod n\}$

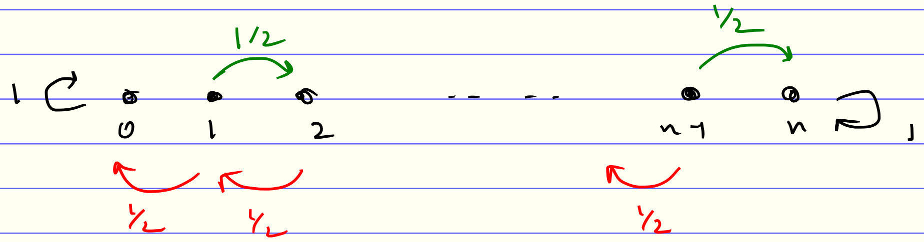
$t > 0$ $\quad D_t$ = clockwise distance from X_t to Y_t.

At time t,



$$\begin{cases} +1 & \frac{p}{2} + \frac{q}{2} = \frac{1}{2} \\ -1 & \frac{p}{2} + \frac{q}{2} = \frac{1}{2} \end{cases}$$

$\text{Range}(D_t) = \{0, 1, \dots, n\}$



D_t performs the simple random walk on $\{1, 2, \dots, n-1\}$ and gets absorbed at 0 or n

$\{D_t\}_{t \geq 0}$ is equivalent Gambler's ruin chain

Define $Z = \min\{t \geq 0 : D_t \in \{0, n\}\}$ = time required by D_t to get absorbed at 0 or n

See that $D_t \in \{0, n\}$ $\forall t \geq s$ is equivalent to saying $\{X_t = Y_t \ \forall t \geq s\}$.

$$Z_{couple} = Z$$

(HW4, Book-Keeping Ques 7) $\mathbb{E}(Z) = k(n-k)$ when given that $D_0 = k$

$t = 0$, $\boxed{D_0 = k}$ represents the clockwise distance from x to y

$$\mathbb{E}_{x,y}(Z_{couple}) = \mathbb{E}(Z) = \underbrace{k(n-k)}_{\leq \frac{n^2}{4}}$$

where k is the clockwise distance from x to y

max occurs $k = \frac{n}{2}$

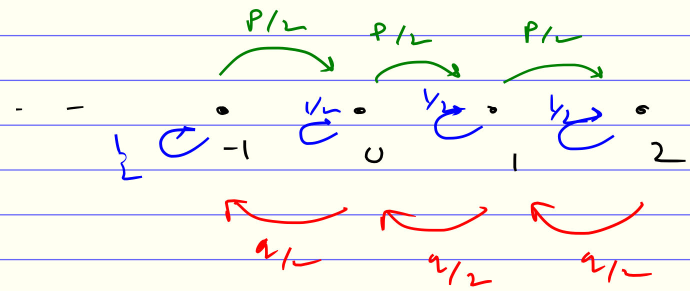
$$\mathbb{E}_{x,y}(Z_{couple}) \leq \frac{n^2}{4}$$

$$t_{mix} \leq 4 \max_{x,y \in \mathbb{Z}_n} \mathbb{E}_{x,y}(Z_{couple}) \leq 4 \cdot \frac{n^2}{4} = n^2$$

∎

Lower Bound:

S_t is a lazy $(p-q)$ biased random walk on $\mathbb{Z}$.



$M_1, M_2, \ldots$ iid $M = \begin{cases} +1 & \text{wp } p/2 \\ -1 & \text{wp } q/2 \\ 0 & \text{wp } 1/2 \end{cases}$

$S_t = \sum_{i=1}^{t} M_i$, $\mu_t := E(S_t) = t(p-q)/2$

$\sigma_t^2 = Var(S_t) = t\left(\frac{1}{4} + pq\right) \le \frac{t}{2}$

max occurs when $p = \frac{1}{2}$

$X_0 = x_0$ $X_t = S_t \bmod n$

$\{X_t\}_{t \ge 0}$ P, $\pi = (\frac{1}{n} : i \in \mathbb{Z}_n)$

$\rho(x,y)$ = distance between x & y on n-cycle

Now for $t \ge 0$, $A_t = \left\{ k : \rho(k, \lfloor x_0 + \mu_t \rfloor \bmod n) \ge \frac{n}{4} \right\}$

$|A_t| \ge \frac{n}{2}$

$\pi(A_t) \ge \frac{1}{2}$

$\mathbb{P}(|S_t - \mu_t| \ge \frac{n}{4}) \le \frac{16\sigma_t^2}{n^2} \le \frac{8t}{n^2}$

$\mathbb{P}(X_t \in A_t \mid X_0 = x_0) = \mathbb{P}(|S_t - \mu_t|) \le \frac{1}{4}$

So for $t < n^2/32$, $\underbrace{|\pi(A_t) - \mathbb{P}(X_t \in A_t \mid X_0 = x_0)|}_{\le d(t)} \ge \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$t < \frac{n^2}{32}$, $d(t) > \frac{1}{4}$.

$\Rightarrow t_{mix} \geq \frac{n^2}{32}$ $\blacksquare$

General Technique: — Hypercube, Torus, finite binary trees.

For the upper bound:

i) Construct the Markovian coupling

ii) Apply Propn 1. to get $t_{mix} \leq 4 \max_{x,y} \mathbb{E}_{x,y}(\tau_{couple})$

iii) Control or bound $\mathbb{E}_{x,y}(\tau_{couple})$ in terms of size of the state space

For the lower bound:

i) Find a sequence of event A_t such that

$$\left| P(X_t \in \cdot \mid X_0 = x) - \pi(A_t) \right| > \frac{1}{4} \quad \forall\, t < a_{|\mathcal{X}|}$$

$\mathcal{X} \to$ state space.

ii) Conclude that $d(t) > \frac{1}{4} \quad \forall\, t < a_{|\mathcal{X}|}$

iii) $t_{mix} > a_{|\mathcal{X}|}$

Sketch of Proof of Proposition 1.

- Compare total variation distance to coupling
- $\| P^t(x,\cdot) - P^t(y,\cdot) \|_{TV} \leq \mathbb{P}_{x,y}(\tau_{couple} > t)$
- $d(t) \leq \max_{x,y \in \mathcal{X}} \mathbb{P}_{x,y}(\tau_{couple} > t)$
- Markov inequality $\Rightarrow$ $t_{mix} \leq 4 \max_{x,y \in \mathcal{X}} \mathbb{E}_{x,y}(\tau_{couple})$