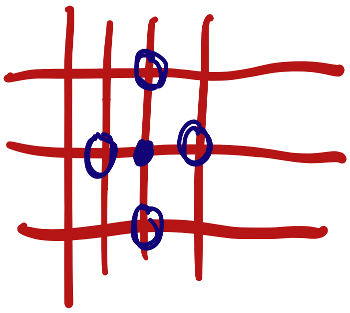
Bounded Harmonic on $\mathbb{Z}^d$

A function $f: \mathbb{Z}^d \to \mathbb{R}$ is **harmonic** if $\forall\, x \in \mathbb{Z}^d$

$$f(x) = \frac{1}{2d}\left(f(x+e_1) + f(x-e_1) + \cdots + f(x+e_d) + f(x-e_d)\right)$$



$\mathbb{Z}^d$ has **Liouville** property if every **bounded** harmonic function is constant.

$\mathbb{Z}^d$ has **strong Liouville** property if every **positive** harmonic function constant.

SLP $\Rightarrow$ LP

Example: $\mathbb{Z}$

Let f be a positive harmonic function

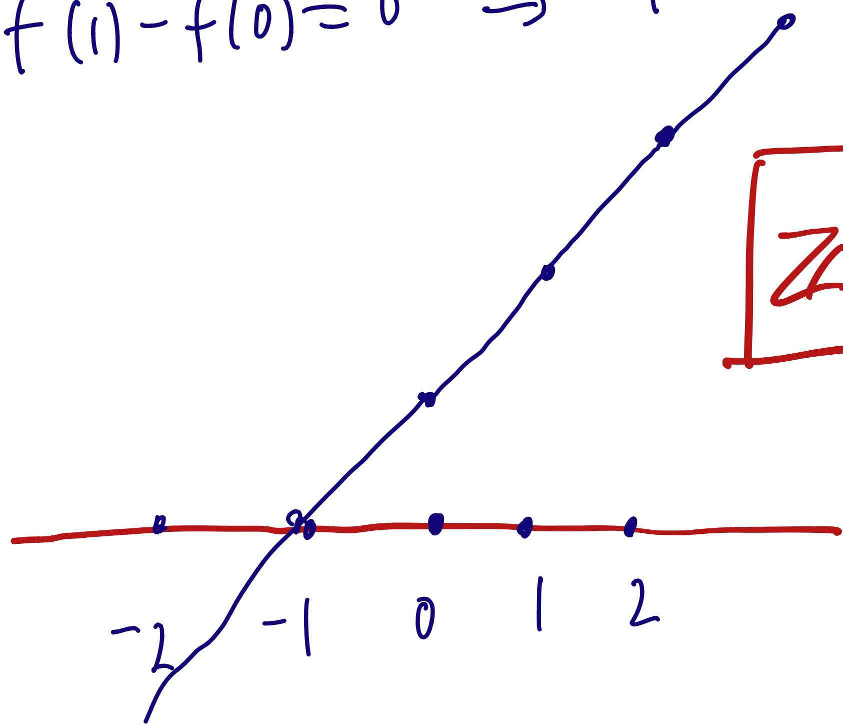
① $f(1) = \dfrac{f(0)+f(2)}{2} \Rightarrow f(2) = f(0) + 2(f(1)-f(0))$

② $f(2) = \dfrac{f(1)+f(3)}{2} \Rightarrow f(3) = f(0) + 3(f(1)-f(0))$

$\vdots$

$\forall x \in \mathbb{Z} \qquad f(x) = f(0) + \cancel{x}(f(1) - f(0))$

$\Rightarrow f(1) - f(0) = 0 \Rightarrow f$ is constant



$\boxed{\mathbb{Z} \text{ has SLP}}$

Questions: ① When does $\mathbb{Z}^d$ have LP?

② When does $\mathbb{Z}^d$ have SLP?

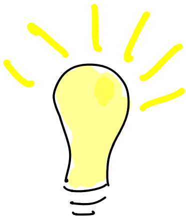
Answer: $\forall\, d \geq 1$ $\mathbb{Z}^d$ has SLP!

What we will prove

	LP	SLP
$\mathbb{Z}^2$	✓	✓
$d>2,\ \mathbb{Z}^d$	✓	

Coupling argument to show sublinear harmonic functions

$$\boxed{\text{Sublinear: } f(x) = o(\|x\|_2)}$$



Harmonic Functions $\longleftrightarrow$ Martingales

$f: \mathbb{Z}^d \to \mathbb{R}$ harmonic

$\{X_n\}_{n \geq 0}$, $\boxed{X_0 = x} \in \mathbb{Z}^d$: SRW

Claim: $\{f(X_n)\}_{n\geq 0}$ is a martingale wrt $\{X_n\}_{n \geq 0}$.

In particular, $\mathbb{E}[f(X_n)] = f(x)$.

Proof: ① $\mathbb{E}[\,|f(X_n)|\,] < \infty \quad \forall\, n$

② $\mathbb{E}[f(X_n) \mid X_{n-1}, \dots, X_0]$

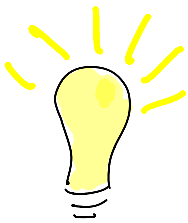
$= \mathbb{E}[f(X_n) \mid X_{n-1}]$

$= \frac{1}{2d}\left[f(X_{n-1}+e_1) + f(X_{n-1}-e_1) + \dots + f(X_{n-1}+e_d) + f(X_{n-1}-e_d)\right]$

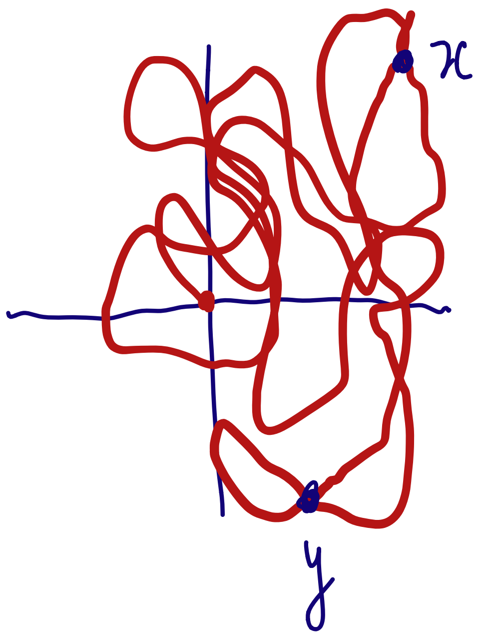
$= f(X_{n-1})$

SLP for $\mathbb{Z}^2$

Let f be a positive harmonic function.



Recurrence and martingale convergence



① $\{X_n\}$ SRW, $X_0 = 0$

② Recurrence $\Rightarrow$ X_n visits x, y ∞-often

③ $f(X_n)$ converges $\Big\}$ positive martingale

$\Rightarrow f(x) = f(y)$

$R = \{ w : X_n(w) \text{ visits } x, y \text{ infinitely often} \}$

$S = \{ w : f(X_n(w)) \text{ converges} \}$

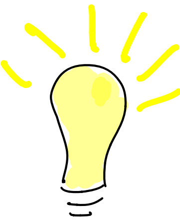
- $\mathbb{P}(R) = 1$, $\mathbb{P}(S) = 1 \Rightarrow \mathbb{P}(S \cap R) = 1 \Rightarrow S \cap R \neq \emptyset$
- Let $w \in S \cap R$. Fix $\epsilon > 0$.
- Since $w \in S \Rightarrow \exists N$ s.t. $|f(X_n(w)) - f(X_m(w))| < \epsilon$ whenever $n, m > N$
- Since $w \in R \Rightarrow \exists N_1, N_2 > N$ s.t. $X_{N_1}(w) = x$ and $X_{N_2}(w) = y$.

Then $$\underbrace{|f(X_{N_1}(w)) - f(X_{N_2}(w))|}_{= |f(x) - f(y)|} < \epsilon$$

$\Rightarrow f(x) = f(y)$

Liouville Property for $\mathbb{Z}^d$

f bounded, harmonic $\overset{?}{\Longrightarrow}$ constant



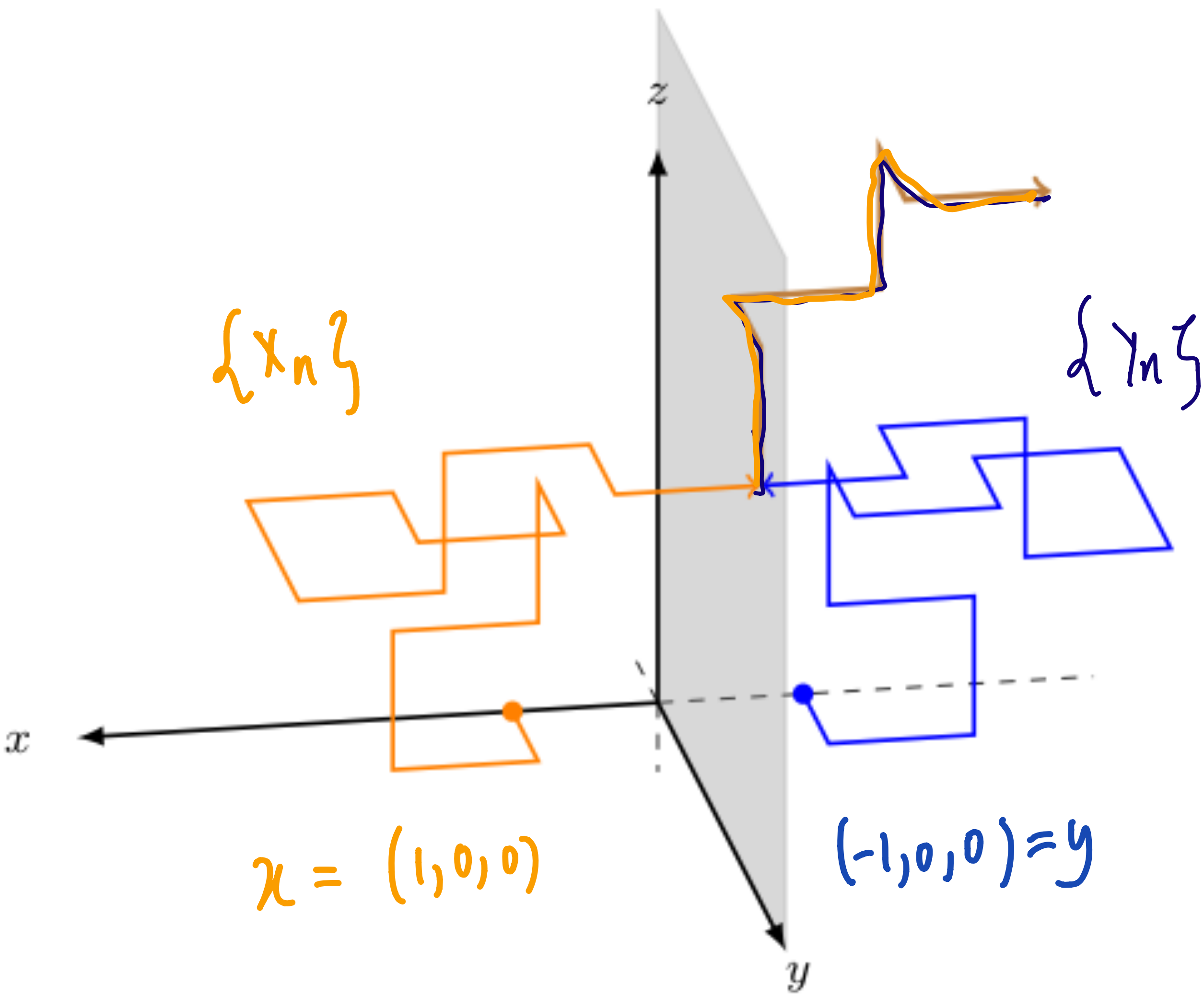
Coupling via reflection +
Martingale convergence

Restrict to $d=3$

$\{X_n\}_{n\geq 0}$, $X_0 = (1,0,0) = x$: SRW

$\{Y_n\}_{n\geq 0}$, $Y_0 = (-1,0,0) = y$: SRW

We will show $f(x) = f(y)$



- $\{X_n\}$, $\{Y_n\}$ SRW
- $X_n(1)$ is a lazy SRW on $\mathbb{Z}$, recurrent
- Meeting has prob. 1

Coupling by Reflection

- $\{X_n = (X_n(1), X_n(2), X_n(3))\}_{n \geq 0}$ SRW on $\mathbb{Z}^3$

  $X_0 = (1,0,0) = x$

- $\tau = \min\{n \geq 0 \mid X_n(1) = 0\}$

- $Y_n = \begin{cases} X_n^{ref} = (-X_n(1), X_n(2), X_n(3)) & \text{if } \tau < n \\ X_n & \text{if } \tau \geq n \end{cases}$

- $\{Y_n\}$ is also a SRW, $Y_0 = (-1,0,0) = y$

- $X_n(1)$ lazy SRW $\Longrightarrow$ recurrent

- So $\mathbb{P}(\tau < \infty) = 1$

  $$\mathbb{P}(\tau < \infty) = \mathbb{P}(\exists N \text{ s.t. } Y_n = X_n \ \forall n \geq N)$$

① $f(X_n), f(Y_n)$ are bounded martingales

$\Rightarrow$ both converge

$$f_X^\infty = \lim f(X_n) \quad , \quad f_Y^\infty = \lim f(Y_n)$$

② $\mathbb{P}(\tau < \infty) = 1 \Rightarrow f_X^\infty = f_Y^\infty$

③ Bounded martingales $\Rightarrow$

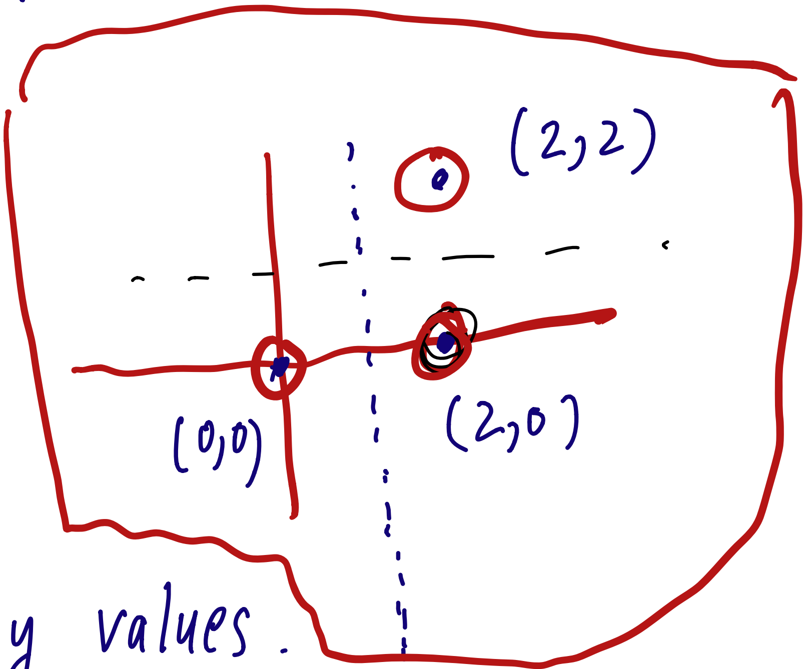
$$\mathbb{E}[f_X^\infty] = \mathbb{E}[f(X_0)] = f(x)$$

$$\mathbb{E}[f_Y^\infty] = \mathbb{E}[f(Y_0)] = f(y)$$

④ $f(x) = f(y)$

• $f(a) = f(b)$ if $a = b \pmod 2$

$a, b \in \mathbb{Z}^3$



• $\Rightarrow$ f takes finitely many values.

$\Rightarrow$ f attains maximum

$\Rightarrow$ f is constant (Exercise)

Blackwell:

$$\boxed{\mathbb{E}\left[f(X_n) - f(Y_n)\right]} = \mathbb{E}\left[\left(f(X_n) - f(Y_n)\right) I(X_n \neq Y_n)\right]$$

$$\leq 2M \, \underset{\downarrow 0}{\mathbb{P}(X_n \neq Y_n)}$$

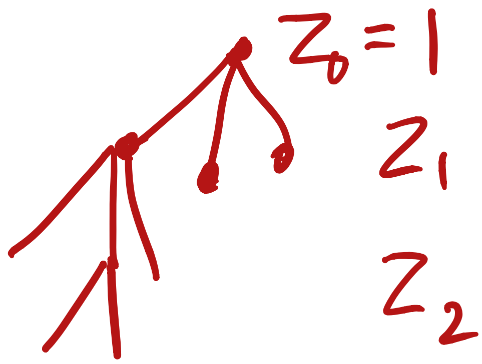
We used boundedness to show

$$\mathbb{E}[f(X_0)] = \mathbb{E}[f_X^{\infty}]$$

Question: $\{Z_n\}_{n \geq 0}$ s.t. $\mathbb{E}[Z_0] \neq \mathbb{E}[Z_\infty]$?

Example: Branching Process example.

$0 < \mu < 1$



- Extinct with prob. 1

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_i^{n+1}$$

iid X