Bounded harmonic functions on $\mathbb{Z}^d$

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1 Introduction

A twice continuously differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be harmonic if $\nabla f = 0$, where ∇ is the Laplacian operator. One of the characterizing properties for harmonic functions on $\mathbb{R}^d$ is the mean-value property. This motivates the following definition of harmonic functions on $\mathbb{Z}^d$.

A function $f : \mathbb{Z}^d \to \mathbb{R}$ is said to be harmonic if for all $x \in \mathbb{Z}^d$,

$$f(x) = \frac{1}{2d}\left(f(x+e_1) + f(x-e_1) + \cdots + f(x+e_d) + f(x-e_d)\right),$$

where $e_1, ..., e_d$ are the standard basis vectors.

We know that every bounded harmonic function on $\mathbb{R}^d$ is a constant function. The analogue of this result for complex differentiable functions is Liouville's theorem, which asserts that every bounded entire function is a constant function. This leads us to the following definitions.

- $\mathbb{Z}^d$ is said to have the **Liouville** property if every bounded harmonic function is a constant function. (A function $f : \mathbb{Z}^d \to \mathbb{R}$ is said to be bounded if there exist $M \in \mathbb{R}$ such that for each $x \in \mathbb{Z}^d$, $|f(x)| < M$.)
- $\mathbb{Z}^d$ is said to have the **strong Liouville** property if every positive harmonic function is a constant function. (A function $f : \mathbb{Z}^d \to \mathbb{R}$ is said to be positive if for each $x \in \mathbb{Z}^d$, $f(x) > 0$.)

Note that if $\mathbb{Z}^d$ has the strong Liouville property then it has the Liouville property, for given a bounded harmonic function we may add a constant function to get a positive harmonic function.

The main result of the subject is that:

Theorem 1. *$\mathbb{Z}^d$ has the strong Liouville property for all $d \geq 1$. [3, page 101]*

We will not prove this theorem. Instead, we prove that $\mathbb{Z}^2$ has strong Liouville property and $\mathbb{Z}^d$ has the Liouville property for all $d \geq 1$. We show that $\mathbb{Z}^2$ has the strong Liouville property using the martingale convergence theorem and recurrence of the simple random walk on $\mathbb{Z}^2$. We establish that $\mathbb{Z}^d$ has the Liouville property using two proofs: one using coupling and the other using reflection and martingale convergence.

We first consider the case $d = 1$ in the example below.

Example: Suppose f is a positive harmonic function on $\mathbb{Z}$. Let $f(0) = a$ and $f(1) = b$. Since f is harmonic, $b = \frac{1}{2}(a + f(2))$ so $f(2) = 2b - a = a + 2(b - a)$. Similarly $a = \frac{1}{2}(f(-1) + b)$ implies $f(-1) = a - (b - a)$. We can show by induction that $f(x) = a + x(b - a)$. Therefore, $f(x)$ cannot be positive for all x unless $a = b$. So we conclude that $f(x) = a$ for all $x \in \mathbb{Z}$. This proves that $\mathbb{Z}$ has the strong Liouville property.

To prove the strong Liouville property for $\mathbb{Z}^2$ and the Liouville property for $\mathbb{Z}^d$, we require the following two propositions about martingale convergence.

Proposition 1. *Suppose $\{X_n\}_{n\geq 0}$ is a martingale bounded in L^1 : $\sup_n \mathrm{E}[|X_n|] < \infty$. Then, with probability 1, $X_\infty = \lim_{n\to\infty} X_n$ exists and is finite. In particular, the conclusion holds if the martingale is positive or bounded. [1]*

Proposition 2. *Suppose $\{X_n\}_{n\geq 0}$ is a bounded martingale. Then $\mathrm{E}[X_\infty] = \mathrm{E}[X_0]$. [5, chapter 14, page 133]*

2 Strong Liouville property for $\mathbb{Z}^2$

In this section we will use the martingale convergence theorem (proposition 1) to show the strong Liouville property for $\mathbb{Z}^2$. We begin with a lemma that gives the connection between harmonic functions and martingales.

Lemma 1. *Let $f : \mathbb{Z}^d \to \mathbb{R}$ be a harmonic function. Let $\{X_n\}_{n\geq 0}$ be the simple random walk on $\mathbb{Z}^d$ with $X_0 = x$ for some $x \in \mathbb{Z}^d$. Then $\{f(X_n)\}_{n\geq 0}$ is a martingale with respect to $\{X_n\}_{n\geq 0}$. In particular, we have $\mathrm{E}[f(X_n)] = f(X_0) = f(x)$.*

Proof. We first fix n and argue that $\mathrm{E}[|f(X_n)|] < \infty$. Since X_n has finite range, $f(X_n)$ does too. In particular, $f(X_n)$ is a bounded random variable, and so $\mathrm{E}[|f(X_n)|] < \infty$. Next we verify the martingale property. For $n > 0$ we have,

$$\begin{aligned}
\mathrm{E}[f(X_n)|X_{n-1}, \ldots, X_0] &\overset{(a)}{=} \mathrm{E}[f(X_n)|X_{n-1}] \\
&\overset{(b)}{=} \frac{1}{2d}\left(f(X_{n-1}+e_1) + f(X_{n-1}-e_1) + \cdots + f(X_{n-1}+e_d) + f(X_{n-1}-e_d)\right) \\
&\overset{(c)}{=} f(X_{n-1}),
\end{aligned}$$

where (a) follows from the Markov property of $\{X_n\}$, (b) follows from the definition of a simple random walk, and (c) follows because f is a harmonic function. $\square$

Proof. Let $f : \mathbb{Z}^2 \to \mathbb{R}$ be a positive harmonic function. Let $\{X_n\}_{n\geq 0}$ be a simple random walk with $X_0 = 0$. Fix $x, y \in \mathbb{Z}^2$. Let

$$\begin{aligned}
R &= \{\omega : X_n(\omega) \text{ visits } x \text{ and } y \text{ infinitely many times}\}; \\
S &= \{\omega : f(X_n(\omega)) \text{ converges}\}.
\end{aligned}$$

Since $\{X_n\}$ is recurrent, we have $\Pr[R] = 1$. Since $\{f(X_n)\}$ is a positive martingale, by proposition 1 it converges with probability 1, and we have $\Pr[S] = 1$. Thus, $\Pr[S \cap R] = 1$; in particular, $S \cap R$ is nonempty. Let $\omega \in S \cap R$ and $y_n = f(X_n(\omega)$. Fix $\epsilon > 0$. Since $\{y_n\}$ converges it is a cauchy sequence. So we can choose $N \in \mathbb{N}$ such that for all $n, m > N$, $|y_n - y_m| < \epsilon$. Next, since $\omega \in R$, we can choose $n', m' > N$ such that $y_{n'} = x$ and $y_{m'} = y$. Therefore we have $|y_{n'} - y_{m'}| < \epsilon$, or $|f(x) - f(y)| < \epsilon$. Since ϵ was arbitrary we conclude that $f(x) = f(y)$. $\square$

3 Liouville property for $\mathbb{Z}^d$

In this section, we give two proofs for the Liouville property for $\mathbb{Z}^d$.

Lemma 2. *Suppose f is harmonic function that attains its supremum. Then it is constant.*

Lemma 3. *Suppose for all $x, y \in \mathbb{Z}^d$ there exists a coupling (X_n, Y_n) such that $\{X_n\}$ and $\{Y_n\}$ are marginally simple lazy random walks, $(X_0, Y_0) = (x, y)$, and*

$$\lim_{n\to\infty} \Pr[X_n \neq Y_n] = 0.$$

Then, every bounded harmonic function on $\mathbb{Z}^d$ is the constant function.

Proof. Let f be a bounded harmonic function. Fix $x, y \in \mathbb{Z}^d$. We will show that $f(x) = f(y)$. Let (X_n, Y_n) be a coupling satisfying the conditions above. Since f is harmonic, $\mathrm{E}[f(X_n)] = f(x)$ and $\mathrm{E}[f(Y_n)] = f(y)$. So for all n, we have $|f(x) - f(y)| = |\mathrm{E}[f(X_n)] - \mathrm{E}[f(Y_n)]|$. Since f is bounded, there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{Z}^d$. Then we have,

$$\begin{aligned}
|f(x) - f(y)| &= |\mathrm{E}[f(X_n)] - \mathrm{E}[f(Y_n)]| \\
&= |\mathrm{E}[(f(X_n) - f(Y_n))\mathrm{I}(X_n \neq Y_n)] \\
&\leq \mathrm{E}[(|f(X_n)| + |f(Y_n)|)\mathrm{I}(X_n \neq Y_n)] \\
&\leq 2M \Pr[X_n \neq Y_n].
\end{aligned}$$

Since $\lim_{n\to\infty} \Pr[X_n \neq Y_n] = 0$, we conclude that $f(x) = f(y)$, as required. □

Theorem 2. *For all $d \geq 1$, $\mathbb{Z}^d$ has the Liouville property.*

We give two proofs.

Proof 1. By lemma 3, it is enough to find a coupling (X_n, Y_n) of two simple lazy random walks X_n and Y_n (with $(X_0, Y_0) = (x, y)$), such that $\Pr[X_n \neq Y_n] \to 0$.

We construct our coupling (X_n, Y_n) as follows. At each step, draw a direction $i \in \{1, \ldots, d\}$ uniformly at random. Then, we have two cases.

$X_n(i) = Y_n(i)$**:** With probability 1/2 leave both walks at the same position; with probability 1/2 move them together in direction i.

$X_n(i) \neq Y_n(i)$**:** Choose either X_n or Y_n with probability 1/2 and move the chosen walk in direction i, leaving the other walk unchanged.

Observe that marginally both walks are distributed as simple lazy random walks. Furthermore, for coordinate i, the difference $X_n(i) - Y_n(i)$ is a lazy simple one-dimensional random walk on $\mathbb{Z}$ (starting at $X_0(i) - Y_0(i) = x(i) - y(i)$), with 0 as an absorbing state. This implies that with probability 1, there exists an N such that for all $n > N$, we have $X_n = Y_n$. Thus, $\lim_{n\to\infty} \Pr[X_n \neq Y_n] = 0$, as required. □

Proof 2. Let $\{X_n\}$ be a simple random walk with $X_0 = (1, 0, \ldots, 0)$. Let $\tau = \min\{n : X_n(1) = 0\}$, that is, the first time at which $X_n(1) = 0$. Define another simple random walk $\{Y_n\}$ (couple with $\{X_n\}$ as follows: if $n < \tau$ then $Y_n = (-X_n(1), X_n(2), \ldots, X_n(d))$, and if $n \geq \tau$ then $Y_n = X_n$. In other words, Y_n is the random walk reflected across the hyperplane $x_1 = 0$ until X_n hits the hyperplane; after that $Y_n = X_n$. Note that $\{Y_n\}$ is marginally a simple random walk starting at $(-1, 0, \ldots, 0)$. Also, 0 is an absorbing state for $X_n(1) - Y_n(1)$ since the simple lazy random walk on $2\mathbb{Z}$ is recurrent. Thus, with probability 1, there exists an N such that for all $n > N$, we have $X_n = Y_n$.

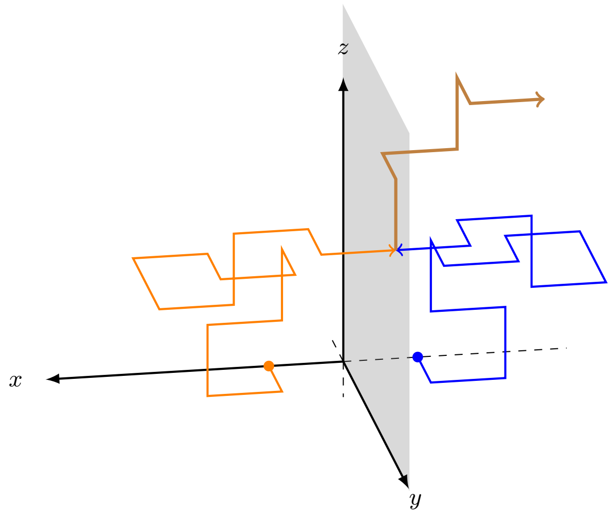

Figure 1: The orange walk represents $\{X_n\}$ and the blue walk represents $\{Y_n\}$. The brown walk represents $\{X_n\}$ and $\{Y_n\}$ after they have met.

Since $\{f(X_n)\}$ and $\{f(Y_n)\}$ are bounded martingales, by proposition 1 they converge. Furthermore if $f_\infty^X = \lim_{n\to\infty} f(X_n)$ and $f_\infty^Y = \lim_{n\to\infty} f(Y_n)$, we have

$$f_\infty^X = f_\infty^Y.$$

So, in particular, we have $\mathrm{E}[f_\infty^X] = \mathrm{E}[f_\infty^Y]$.

Now, since $f(X_n)$ and $f(Y_n)$ are bounded martingales, by proposition 2 we have,

$$\mathrm{E}[f(X_0)] = \mathrm{E}[f_\infty^X] \text{ and } \mathrm{E}[(f(Y_0)] = \mathrm{E}[f_\infty^Y].$$

So we get

$$f(X_0) = \mathrm{E}[f(X_0)] = \mathrm{E}[f_\infty^X] = \mathrm{E}[f_\infty^Y] = \mathrm{E}[f(Y_0)] = f(Y_0).$$

Thus, $f((1,0,\dots,0) = f((-1,0,\dots,0))$. By shifting the origin to a point $x \in \mathbb{Z}^d$ and repeating the above argument in direction i, we conclude that $f(x+e_i) = f(x-e_i)$. This implies that $f(x) = f(y)$ whenever $x = y \pmod 2$. Therefore f takes at most 2^d different values, so f attains its supremum. By lemma 2 f is a constant function.

$\square$

4 Other techniques

Another proof of the Liouville property

Let $h : \mathbb{Z}^d \to \mathbb{R}$ be a bounded harmonic function. Here are the steps for another method to show that h is constant. Let $Y_1, Y_2, \dots$ be iid Y, where $Y = e_i$ with probability $1/2d$ and $Y = -e_i$ with probability $1/2d$, for each $1 \le i \le d$.

1. Fix $x \in \mathbb{Z}^d$. Consider the sequence defined by

$$u_1 = \mathrm{E}_{Y_1}[(h(x+Y_1) - h(x))^2],$$

and for $n \ge 2$,

$$u_n = \mathrm{E}_{Y_1,\dots,Y_n}[(h(x+Y_1+\dots+Y_n) - h(x+Y_2+\dots+Y_n))^2].$$

Show by the Cauchy-Schwarz inequality that $u_n \ge u_{n-1}$.

2. Show that

$$u_n = \mathrm{E}_{Y_1,\dots,Y_n}[h^2(x+Y_1+\dots+Y_n)] - \mathrm{E}_{Y_1,\dots,Y_{n-1}}[h^2(x+Y_1+\dots+Y_{n-1})].$$

3. Show that $\sum_{n\ge1} u_n$ is a converging series of increasing non-negative terms. Conclude that $u_1 = 0$.

4. Conclude that h is a constant function.

For this argument given in a more general context see the introduction of [4].

Extension of the coupling argument to sub-linear functions

A function $f : \mathbb{Z}^d \to \mathbb{R}$ is said to be sub-linear if $f(x) = o(\|x\|)$, where $\|x\|$ is the euclidean norm of x. Suppose f is a sub-linear, harmonic function. Use the reflection coupling and show that $\mathrm{E}[f(X_n) - f(Y_n)] \to 0$. Conclude that f is a constant function.

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References

[1] Siva Athreya. Class Notes, Topics in Applied Stochastic Processes.
`https://www.isibang.ac.in/~athreya/Teaching/tas/PDF/apr13.pdf`

[2] Yuval Peres. Aspects of Markov Chains (Slides).
`https://services.math.duke.edu/~rtd/CPSS2009/peres6.pdf`

[3] Gábor Pete. Probability and Geometry on Groups.
`https://math.bme.hu/~gabor/PGG.pdf`

[4] Albert Raugi. A general Choquet–Deny theorem for nilpotent groups. http://www.numdam.org/article/AIHPB_2004__40_6_677_0.pdf

[5] David Williams. Probability with Martingales, Cambridge.