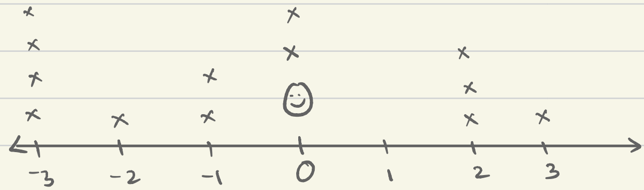
Survival Probability in a Poisson field of moving traps.

Setup and Notation.

$\mathbb{Z}^d$, say $d=1$



$X : \mathbb{N} \cup \{0\} \to \mathbb{Z}^d$, $X(0) = 0$. (you.)

N_y = # of traps at $y \in \mathbb{Z}^d$ at time 0. $N_y \sim \text{Pois}(\lambda)$

$\{Y_j^y\}_{y \in \mathbb{Z}^d,\, 1 \le j \le N_y}$ is a family of lazy symmetric RWs that are ind of each other

Y_j^y is the j^{th} RW starting at $y \in \mathbb{Z}^d$.

$0 < q < 1$ Trapping probability. If $X(n) = Y_j^y(n)$ wp q X gets trapped,

$1-q$ it escapes.

$$\xi(n,x) = \text{\# of trap RWs at } x \in \mathbb{Z}^d \text{ at time } n$$

$$= \sum_{y \in \mathbb{Z}^d,\, 1 \le j \le N_y} \mathbb{1}\{Y_j^y(n) = x\}$$

$(1-q)^{\xi(i, X(i))}$. is the prob that X survives time i given it survives time $i-1$.

$(1-q)^{\sum_{i=0}^{n} \xi(i, X(i))}$. is the prob that X survives upto time n.

$\sigma^X(n)$. = The Annealed / Average Survival Probability $= E^{\xi}\left[(1-q)^{\sum_{i=0}^{n} \xi(i, X(i))}\right]$

<u>Theorem</u>: Let $\xi, N_y, \{Y_j^y\}$ be as above. Then,

$$\sigma^0(n) \geq \sigma^X(n) \quad \forall n$$

where $\sigma^0(n)$ denotes the survival prob of the trajectory that remains at 0 indefinitely.

Overview of the Proof

Lemma 1: Let $X: \mathbb{N} \cup \{0\} \to \mathbb{Z}^d$, $X(0) = 0$. $\quad \sigma^X(n) = \exp\{-\lambda \sum_{y \in \mathbb{Z}^d} \omega^X(n,y)\}$

where $\quad \omega^X(n,y) = E_y^Y[\, 1 - (1-q)^{\sum_{i=0}^{n} \mathbb{1}\{Y(i) = X(i)\}}\,]$.

$\omega^X(n,y)$ = The prob that X gets captured by a trap Y starting at $y \in \mathbb{Z}^d$ in the first n steps with the average taken over all the realizations of Y.

Lemma 2:
$$\sum_{y \in \mathbb{Z}^d} \omega^X(n,y) \geq \sum_{y \in \mathbb{Z}^d} \omega^0(n,y).$$

If you have L1 and L2,

$$\exp\{-\lambda \sum_{y \in \mathbb{Z}^d} \omega^X(n,y)\} \leq \exp\{-\lambda \sum_{y \in \mathbb{Z}^d} \omega^0(n,y)\}$$

$$\overset{L1}{\Longrightarrow} \quad \sigma^X(n) \leq \sigma^0(n).$$

$\square$

$\omega^X(n,y)$

Define a stopping time to be when a walk X gets captured by Y.
$= \tau_X$

1. $X(n) = Y(n)$
2. Trap at Y is open. $\{Z_i = 1\}$ = Trap Y is open at time i
   $\{Z_i = 0\}$ = " closed "

$\tau_x = \min\{ i \geq 0 : X(i) = Y(i),\ Z_i = 1 \}$

starting at $y \in \mathbb{Z}^d$

$P^Y_y(\tau_x \leq n)$ is the prob that X has been captured by Y by time n.

<u>Lemma 2.1</u> $\quad w^x(n,y) = P^Y_y(\tau_x \leq n)$

Let's assume L2.1., L2 is reduced to

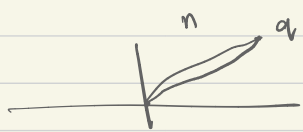
$$\sum_{y \in \mathbb{Z}^d} P^Y_y(\tau_x \leq n) \geq \sum_{y \in \mathbb{Z}^d} P^Y_y(\tau_0 \leq n)$$

Let's look at q, the trapping probability.

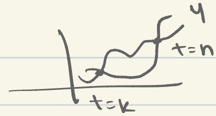
$$P(Z_i = 1) = q \qquad \forall i$$

$$P^Y_{X(n)}\Big(\bigcup_{y \in \mathbb{Z}^d} \{Y(n) = y\}\Big) = 1$$

$$\begin{aligned} q &= P^Y_{X(n)}\Big(\bigcup_{y \in \mathbb{Z}^d} \{Y(n) = y\},\ Z_n = 1\Big) \\ &= \sum_{y \in \mathbb{Z}^d} P^Y_{X(n)}(Y(n) = y,\ Z_n = 1) \\ &= \sum_{y \in \mathbb{Z}^d} P^Y_y(Y(n) = X(n),\ Z_n = 1) \end{aligned}$$



Let $p^Y_n(x)$ = the prob that Y starting at 0 reaches $x \in \mathbb{Z}^d$ in n steps.



$$q = \sum_{y\in\mathbb{Z}^d} P_y^Y\left(Y(n)=X(n),\ Z_n=1\right)$$

(1) $\rightarrow$ $$q = \sum_{y\in\mathbb{Z}^d} P_y^Y(\tau_x=n) + \sum_{y\in\mathbb{Z}^d}\sum_{k=0}^{n-1} P_y^Y(\tau_x=k)\times p_{n-k}^Y\left(X(n)-X(k)\right)\times q$$

(2) $\rightarrow$ $$q = \sum_{y\in\mathbb{Z}^d} P_y^Y(\tau_0=n) + \sum_{y\in\mathbb{Z}^d}\sum_{k=0}^{n-1} P_y^Y(\tau_0=k)\, p_{n-k}^Y(0)\, q$$

Lemma 2.2: $$p_n^Y(y) \le p_n^Y(0) \qquad \forall\, n, y$$
$$p_{n+1}^Y(0) \le p_n^Y(0)$$

$\Downarrow$ using (1), (2), Lemma 2.2

$$\sum_{y\in\mathbb{Z}^d} P_y^Y(\tau_x\le n) \ge \sum_{y\in\mathbb{Z}^d} P_y^Y(\tau_0\le n).$$

$\square$