Survival Probability in a Poisson field of moving traps.

Section A : Introduction

Let's say you're at the origin in the integer lattice. Along with you, there are traps distributed throughout the lattice, each of which perform a random walk. If you intersect with a trap, you either get captured, say with some probability q, or you escape. Suppose at each point, a poisson number of traps are placed at time 0. How would you move so as to maximise your Survival probability?

We shall try to understand the best survival strategy formally. Assume there is a poisson field of traps on $\mathbb{Z}^d$, each following a symmetric random walk. Let X denote a particular trajectory that you choose and let Y denote a particular trap random walk. At some time n, if $X(n) = Y(n)$ then X gets captured wp q and escapes wp $1-q$. Each of the random walks in the poisson field try to capture X in the above manner.

In this note, we try to understand the survival probability of x averaged over the poisson field of moving traps. Specifically, we show that this average survival probability is upper bounded by the average survival probability of the trajectory that remains at 0. This is called the Pascal Principle because Pascal once asserted:

" All misfortune of men comes from the fact that he doesn't stay peacefully in his room"

Section B : The Setup.

In this section, we present the required notation and the setup:

$\{X(n)\}_{n\geq 0}$ is a map $X: \mathbb{N}\cup\{0\} \to \mathbb{Z}^d$, with $X(0)=0$.

$\{N_y\}_{y\in\mathbb{Z}^d}$, where $N_y \overset{iid}{\sim}$ Poisson(λ) for $y\in\mathbb{Z}^d$ denotes the number of traps at each point $y\in\mathbb{Z}^d$. This is the Poisson field of moving traps.

$\{Y_j^y\}_{y\in\mathbb{Z}^d, 1\leq j\leq N_y}$ is a family of lazy independent symmetric random walks on $\mathbb{Z}^d$. For a particular $y\in\mathbb{Z}^d$ and $1\leq j\leq N_y$, Y_j^y denotes the j^{th} walk starting at $y\in\mathbb{Z}^d$.

The trapping probability, is denoted by $0<q<1$. This means that if $X(n) = Y_j^y(n)$ at some time n, for some $y\in\mathbb{Z}^d$ and $1\leq j\leq N_y$ then the trap walk Y_j^y captures X with probability q and X survives with probability $1-q$, (independent of y, j.)

Let $\xi(n,x)$ denote the number of traps at time n at position $x\in\mathbb{Z}^d$. Formally,

$$\xi(n,x) = \sum_{y\in\mathbb{Z}^d,\, 1\leq j\leq N_y} \delta_x(Y_j^y(n))$$

where $\delta_x(y) = \begin{cases} 1 & \text{if } x=y, \quad 0 \text{ if } x\neq y. \end{cases}$

Let's assume that the trajectory X has survived upto time n. Now, for it to survive at time n, it must not be captured by any of the walks at $X(n)$ at time n. Observe that $\xi(n, X(n))$ is the number of traps at $X(n)$ at time n. Now, the probability that X survives is $(1-q)^{\xi(n,X(n))}$. Hence, the probability that X is captured at time n is $1-(1-q)^{\xi(n,X(n))}$.

The probability that X survives upto time n is $(1-q)^{\sum_{i=0}^{n} \xi(i, X(i))}$.

We want to study the average survival probability, denoted by $\sigma^X(n)$, of the walk X with the average taken over the the Poisson field of traps.

$$\sigma^X(n) = E^{\xi}[(1-q)^{\sum_{i=0}^{n} \xi(i, X(i))}].$$

The notation E^{ξ} means you're taking the expectation over ξ.

We will show that $\sigma^X(n)$ is maximized for $X=0$. This is our main result that is stated in the next section. This treatment has been taken from Lemma 2.2 of [1].

Section C: The result and an overview of the proof.

Theorem 1: Let $X: \mathbb{N} \cup \{0\} \to \mathbb{Z}^d$ be any given trajectory, where $X(0)=0$. For a Poisson System of moving traps, let $\sigma^X(n)$ denote the average survival probability of the walk X upto n steps. Then,

$$\sigma^0(n) \geq \sigma^X(n) \qquad \forall n \in \mathbb{N}$$

where $\sigma^0(n)$ denotes the survival probability of a trajectory that uniformly remains at 0. (i.e. $\sigma^0(n) = E^{\xi}[(1-q)^{\sum_{i=1}^{n} \xi(i,0)}]$.)

Proof: We know that for any $X: \mathbb{N} \cup \{0\} \to \mathbb{Z}^d$, the survival probability is

$$\sigma^X(n) = E^{\xi}[(1-q)^{\sum_{i=0}^{n} \xi(i, X(i))}].$$

First we average out the Poisson Field, this is made precise in Lemma 1.

Lemma 1: Let $X: \mathbb{N} \cup \{0\} \to \mathbb{Z}^d$, $X(0)=0$ be given, $\sigma^X(n) = \exp\{-\lambda \sum_{y \in \mathbb{Z}^d} w^X(n,y)\}$

where

$$w^X(n,y) = E_y^Y[1-(1-q)^{\sum_{i=1}^{n} \mathbb{1}\{Y(i)=X(i)\}}] = 1 - E_y^Y[(1-q)^{\sum_{i=1}^{n} \mathbb{1}\{Y(i)=X(i)\}}].$$

Note that in the above lemma, $\omega^x(n,y)$ is the probababability that x gets captured by a trap Y starting at $y \in \mathbb{Z}^d$ in the first n steps with the average taken over Y.

Given Lemma 1, we will show $\sum_{y \in \mathbb{Z}^d} \omega^x(n,y) \geq \sum_{y \in \mathbb{Z}^d} \omega^0(n,y)$ implying the result.

Now, to better understand $\omega^x(n,y)$, we define a stopping time for when x gets captured by Y. For x to get captured by Y, we would require:

(a) The trajectory x and the walk Y must coincide and

(b) The trap at Y must be open, i.e., the trap Y may capture x, an event happening independantly with probability q.

Let $Z_i \sim Ber(q)$ denote the state of the trap of Y at time i.
If $Z_i = 1$, the trap is open and if $Z_i = 0$, the trap is closed at time i.

Let τ_x denote the stopping time mentioned above. Then,

$$\tau_x = \min\{ i \geq 0 : Y(i) = x(i), Z_i = 1 \}. \qquad \text{(Eq 2.1)}$$

Intuitively, one might think of $P(\tau_x \leq n)$ to denote the probability that x has been captured by Y by time n.

Lemma 2: In the setup described above, $\omega^x(n,y) = P_y^Y(\tau_x \leq n)$.

Using the above lemma, it's easy to complete the proof if we can show the following result.

Lemma 3: For the above setup, $\sum_{y \in \mathbb{Z}^d} P_y^Y(\tau_x \leq n) \geq \sum_{y \in \mathbb{Z}^d} P_y^Y(\tau_0 \leq n)$.

If we have Lemma 3, then we're done with the proof because

$$\sum_{y\in\mathbb{Z}^d} P_y^Y(\tau_x \le n) \geq \sum_{y\in\mathbb{Z}^d} P_y^Y(\tau_0 \le n)$$

$$\Rightarrow \text{(Lemma 2)} \qquad \sum_{y\in\mathbb{Z}^d} \omega^x(n,y) \geq \sum_{y\in\mathbb{Z}^d} \omega^0(n,y)$$

$$\Rightarrow \qquad \exp\{-\lambda \sum_{y\in\mathbb{Z}^d} \omega^x(n,y)\} \le \exp\{-\lambda\sum_{y\in\mathbb{Z}^d} \omega^0(n,y)\}$$

$$\Rightarrow \text{(Lemma 1)} \qquad \sigma^x(n) \le \sigma^0(n). \qquad \square$$

Therefore, the probability that any trajectory x surviving upto time n on average cannot be more than the probability of survival given you remain at the origin. Pascal Principle has been formalized!

Section D: Proof of Lemmas 1,2,3

Lemma 1: Let $x:\mathbb{N}\cup\{0\}\to\mathbb{Z}^d$, $x(0)=0$ be given, $\sigma^x(n)=\exp\{-\lambda\sum_{y\in\mathbb{Z}^d}\omega^x(n,y)\}$ where

$$\omega^x(n,y) = E_y^Y\left[1-(1-q)^{\sum_{i=1}^n \mathbb{1}\{Y(i)=x(i)\}}\right] = 1-E_y^Y\left[(1-q)^{\sum_{i=1}^n \mathbb{1}\{Y(i)=x(i)\}}\right].$$

Proof: (This formal proof isn't rigorous as it involves interchanging of sums and interchange of countable products and expectations.)

We know

$$\sigma^x(n) = E^{\xi}\left[(1-q)^{\sum_{i=0}^n \xi(i,x(i))}\right] = E^{\xi}\left[(1-q)^{\sum_{i=0}^n \sum_{y\in\mathbb{Z}^d}\sum_{j=1}^{N_y} \delta_{x(i)}(Y_j^y(i))}\right]$$

$$= E^{\xi}\left[\prod_{y\in\mathbb{Z}^d}\prod_{1\le j\le N_y}(1-q)^{\sum_{i=0}^n \delta_{x(i)}(Y_j^y(i))}\right]$$

Since these walks Y are independent of each other and the Poisson field,

$$\sigma^x(n) = \prod_{y\in\mathbb{Z}^d} E^{\xi}\Big[\prod_{1\le j\le N_y} (1-q)^{\sum_{i=0}^n \delta_{x(i)}(Y_j^y(i))}\Big].$$

$$= \prod_{y\in\mathbb{Z}^d} E_y^Y E^{N_y}\Big[\prod_{1\le j\le N_y} (1-q)^{\sum_{i=0}^n \delta_{x(i)}(Y_j^y(i))}\Big].$$

As $N_y \overset{iid}{\sim} \text{Pois}(\lambda)$, we take this mean to get

$$\sigma^x(n) = \prod_{y\in\mathbb{Z}^d} E_y^Y\Big[\sum_{k=0}^{\infty} P(N_y=k)\prod_{1\le j\le k} (1-q)^{\sum_{i=0}^n \delta_{x(i)}(Y_j^y(i))}\Big]$$

$$= \prod_{y\in\mathbb{Z}^d} E_y^Y\Big[\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!}\prod_{1\le j\le k} (1-q)^{\sum_{i=0}^n \delta_{x(i)}(Y_j^y(i))}\Big]$$

$$= \prod_{y\in\mathbb{Z}^d} \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} E_y^Y\Big[\prod_{1\le j\le k} (1-q)^{\sum_{i=0}^n \delta_{x(i)}(Y_j^y(i))}\Big]$$

$$= \prod_{y\in\mathbb{Z}^d} \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \left(E_y^Y\Big[(1-q)^{\sum_{i=0}^n \delta_{x(i)}(Y_j^y(i))}\Big]\right)^k$$

The last equation holds because $(1-q)^{\sum_{i=0}^n \delta_{x(i)}(Y_j^y(i))}$ is iid for each $1\le j\le N_y$.
Now, we have

$$\sigma^x(n) = \prod_{y\in\mathbb{Z}^d} \sum_{k=0}^{\infty} \frac{e^{-\lambda}}{k!} \left(\lambda E_y^Y\Big[(1-q)^{\sum_{i=0}^n \delta_{x(i)}(Y_j^y(i))}\Big]\right)^k$$

$$= \prod_{y\in\mathbb{Z}^d} e^{-\lambda} \times \exp\Big\{\lambda E_y^Y\Big[(1-q)^{\sum_{i=0}^n \delta_{x(i)}(Y_j^y(i))}\Big]\Big\}$$

$$= \prod_{y\in\mathbb{Z}^d} \exp\Big\{-\lambda\Big(1 - E_y^Y\Big[(1-q)^{\sum_{i=0}^n \delta_{x(i)}(Y_j^y(i))}\Big]\Big)\Big\}$$

$$= \prod_{y\in\mathbb{Z}^d} \exp\Big\{-\lambda\Big(1 - E_y^Y\Big[(1-q)^{\sum_{i=0}^n \mathbb{1}\{x(i)=Y_j^y(i)\}}\Big]\Big)\Big\}$$

$$= \prod_{y\in\mathbb{Z}^d} \exp\{-\lambda\, \omega^x(n,y)\} = \exp\Big\{-\lambda \sum_{y\in\mathbb{Z}^d} \omega^x(n,y)\Big\}.$$

This completes the proof of Lemma 1. $\square$

Lemma 2: In the setup described above, $w^x(n,y) = P_y^Y(\tau_x \leq n)$.

Proof: Observe that $P_y^Y(\tau_x \leq n) = 1 - P_y^Y(\tau_x > n)$

$$= 1 - P_y^Y\left(\bigcap_{i=0}^{n} (X(i) = Y(i), Z_i = 1)^c\right)$$

$$P_y^Y(\tau_x \leq n) = 1 - P_y^Y\left(\bigcap_{i=0}^{n} \left(\{X(i) \neq Y(i)\} \cup \{Z_i = 0\}\right)\right) \quad \text{(Eq 3.1)}$$

Let A be the event given by $A = \bigcap_{i=0}^{n} \left(\{X(i) \neq Y(i)\} \cup \{Z_i = 0\}\right)$. Then,

$$\mathbb{1}\{A\} = \mathbb{1}\left\{\bigcap_{i=0}^{n} \left(\{X(i) \neq Y(i)\} \cup \{Z_i = 0\}\right)\right\} = \prod_{i=0}^{n} \mathbb{1}\{\{X(i) \neq Y(i)\} \cup \{Z_i = 0\}\}.$$

$$= \prod_{i=0}^{n} \mathbb{1}\{Z_i = 0\}^{\mathbb{1}\{(X(i) \neq Y(i))^c\}} = \prod_{i=0}^{n} \mathbb{1}\{Z_i = 0\}^{\mathbb{1}\{X(i) = Y(i)\}}$$

Replacing this value in Equation 3.1,

$$P_y^Y(\tau_x \leq n) = 1 - E\left[\prod_{i=0}^{n} \mathbb{1}\{Z_i = 0\}^{\mathbb{1}\{X(i) = Y(i)\}}\right]$$

$$= 1 - E_y^Y\left[(1-q)^{\sum_{i=0}^{n} \mathbb{1}\{X(i) = Y(i)\}}\right]$$

The last equation here holds because Z_i is independent of the event $\{X(i) = Y(i)\}$ and $P(Z_i = 0) = (1-q)$.

This completes the proof of Lemma 2. $\square$

Lemma 3: For the above setup, $\sum_{y\in\mathbb{Z}^d} P_y^Y(\tau_x \le n) \geq \sum_{y\in\mathbb{Z}^d} P_y^Y(\tau_0 \le n)$.

Proof: Before we get to the proof of this lemma, another small result is needed.

Lemma 3.1: For a lazy symmetric mean 0 walk Y, let $p_n^Y(y)$ denote the probability of reaching $y\in\mathbb{Z}^d$ in n steps from 0. Then,

$$p_n^Y(0) \geq p_n^Y(y) \quad \forall\ n\geq 0,\ y\in\mathbb{Z}^d \quad \text{and} \quad p_n^Y(0) \geq p_{n+1}^Y(0) \quad \forall\ n\geq 0.$$

The proof of Lemma 3.1 is left as an exercise to the reader. (Hint: Use the Fourier inversion of the characteristic function to get $p_n^Y(y) = \int (\phi(\theta))^n e^{i\theta y}\, d\theta$.)

Proof of Lemma 3: First we try to understand the trapping probability q in the following way. We know $P(Z_i=1) = q \quad \forall i$.

If the random walk Y starts at $X(n)$, then after time n, we have

$$P_{X(n)}^Y\left(\bigcup_{y\in\mathbb{Z}^d}\{Y(n)=y\}\right) = 1$$

Since we have $P(Z_n=1)=q$, combining this in the above equation

$$q = P_{X(n)}^Y\left(\bigcup_{y\in\mathbb{Z}^d}\{Y(n)=y\},\ Z_n=1\right)$$

$$\Rightarrow \quad q = \sum_{y\in\mathbb{Z}^d} P_{X(n)}^Y(Y(n)=y,\ Z_n=1).$$

Since the walk Y is symmetric, it's time reversible. Therefore,

$$q = \sum_{y\in\mathbb{Z}^d} P_y^Y(Y(n)=X(n),\ Z_n=1).$$

The term in the summation, $P_y^Y(Y(n)=X(n), Z_n=1)$, denotes the probability that a walk Y starting at $y\in\mathbb{Z}^d$ collides with X at time n when its trap is open.

Now, at time n, the walk X may get captured implying that the stopping time τ_X is n or, the stopping time τ_X is some $k<n$.

$$q = \sum_{y \in \mathbb{Z}^d} P_y^Y(Y(n)=X(n), Z_n=1)$$

$$= \sum_{y \in \mathbb{Z}^d} P_y^Y(\tau_X=n) + \sum_{y \in \mathbb{Z}^d} \sum_{k=0}^{n-1} P_y^Y(\tau_X=k) \times p_{n-k}^Y(X(n)-X(k)) \times q.$$

Using Lemma 3.1,

$$q \le \sum_{y \in \mathbb{Z}^d} P_y^Y(\tau_X=n) + \sum_{y \in \mathbb{Z}^d} \sum_{k=0}^{n-1} P_y^Y(\tau_X=k) \times p_{n-k}^Y(0) \times q. \quad \text{(Eq 3.2)}$$

Now we apply the same arguement for $X \equiv 0$ to get as $P_0^Y(\bigcup_{y \in \mathbb{Z}^d} Y(n)=y)=1$, and $P(Z_n=1)=q$.

$$q = P_0^Y(\bigcup_{y \in \mathbb{Z}^d} Y(n)=y, Z_n=1) = \sum_{y \in \mathbb{Z}^d} P_0^Y(Y(n)=y, Z_n=1)$$

$$q = \sum_{y \in \mathbb{Z}^d} P_y^Y(Y(n)=0, Z_n=1)$$

$$q = \sum_{y \in \mathbb{Z}^d} P_y^Y(\tau_X=n) + \sum_{y \in \mathbb{Z}^d} \sum_{k=0}^{n-1} P_y^Y(\tau_X=k) \times p_{n-k}^Y(0) \times q. \quad \text{(Eq 3.3)}$$

Comparing Equation 3.2 and 3.3, we get

$$\sum_{y \in \mathbb{Z}^d} P_y^Y(\tau_X=n) + \sum_{y \in \mathbb{Z}^d} \sum_{k=0}^{n-1} P_y^Y(\tau_X=k) p_{n-k}^Y(0) q \le \sum_{y \in \mathbb{Z}^d} P_y^Y(\tau_X=n) + \sum_{y \in \mathbb{Z}^d} \sum_{k=0}^{n-1} P_y^Y(\tau_X=k) p_{n-k}^Y(0) q \quad \text{(Eq 3.4)}$$

Now we define S_n^X to be the probability that X is captured by Y on or before time n and S_n^0 to be the same for a walk that remains at 0.

$$S_n^X = \sum_{y \in \mathbb{Z}^d} P_y^Y(\tau_X \le n) \quad \text{and} \quad S_n^0 = \sum_{y \in \mathbb{Z}^d} P_y^Y(\tau_0 \le n). \quad \text{(Eq 3.5)}$$

It can be observed that,

$$S_0^x = \sum_{y\in\mathbb{Z}^d} P_y^Y(\tau_x = 0) = \sum_{y\in\mathbb{Z}^d} P_y^Y(Y(0) = x(0), Z_0 = 1) = \sum_{y\in\mathbb{Z}^d} P_y^Y(Y(0) = 0, Z_0 = 1)$$

$$= P_0^Y(Y(0) = 0, Z_0 = 1) = P(Z_0 = 1) = q$$

Using a similar arguement it can be shown that $S_0^o = q$.

Now we replace the values from Equation 3.5 in Equation 3.4 to get,

$$(S_n^o - S_{n-1}^o) + q\sum_{k=0}^{n-1} p_{n-k}^Y(0)(S_k^o - S_{k-1}^o) \leq (S_n^x - S_{n-1}^x) + q\sum_{k=0}^{n-1} p_{n-k}^Y(0)(S_k^x - S_{k-1}^x)$$

$$\Rightarrow q\sum_{k=1}^{n-1} p_{n-k}^Y(0)(S_k^o - S_{k-1}^o - S_k^x + S_{k-1}^x) + (S_{n-1}^x - S_{n-1}^o) \leq S_n^x - S_n^o$$

$$\Rightarrow q\sum_{k=1}^{n-1} p_{n-k}^Y(0)(S_k^o - S_k^x) - q\sum_{k=1}^{n-2} p_{n-k+1}^Y(0)(S_k^o - S_k^x) + S_{n-1}^x - S_{n-1}^o \leq S_n^x - S_n^o$$

$$\Rightarrow q p_1^Y(0)(S_{n-1}^o - S_{n-1}^x) + q\sum_{k=1}^{n-2}(p_{n-k}^Y(0) - p_{n-k+1}^Y(0))(S_k^o - S_k^x) + (S_{n-1}^x - S_{n-1}^o) \leq S_n^x - S_n^o$$

$$\Rightarrow S_n^x - S_n^o \geq (S_{n-1}^x - S_{n-1}^o)(1 - q p_1^Y(0)) + q\sum_{k=1}^{n-2}(p_{n-k}^Y(0) - p_{n-k+1}^Y(0))(S_k^o - S_k^x)$$

We know $S_0^x - S_0^o = 0$, $p_{n-k}^Y(0) \leq p_{n-k+1}^Y(0)$ from Lemma 3.1 and $1 - q p_1^Y(0) > 0$. This implies that the term on the right will always be positive. Hence, $S_n^x > S_n^o \quad \forall n$

$$\Rightarrow \sum_{y\in\mathbb{Z}^d} P_y^Y(\tau_x \leq n) \geq \sum_{y\in\mathbb{Z}^d} P_y^Y(\tau_0 \leq n).$$

This completes the proof of Lemma 3. □

References

[1] Survival Probability of a Random walk Among a Poisson System of Moving traps – Alexander Drewitz, Jürgen Gärtner, Alejandro F Ramírez Rongfeng Sun. (arXiv:1010.3958).