Recall :-

- $\{S_n\}_{n\geq 1}$, $S_0 = 0$ $\qquad S_n = \sum_{i=1}^{n} X_i$, $X_i(\omega) = \omega_i = \begin{cases} +1 & \text{wp } \frac{1}{2} \\ -1 & \text{wp } \frac{1}{2} \end{cases}$

$1 \leq n \leq N$ $\qquad \Omega = \{-1,1\}^N$, $\quad \mathbb{P}(A) = \dfrac{|A|}{2^N}$ $\quad A \subseteq \Omega$.

- Stopping time $\quad T : \Omega \to \{0,1,\dots,N\} \cup \{\infty\}$

$\{T=n\} \in \mathcal{A}_n \equiv$ sets observable by time n.

$:= \{(X_1,\dots,X_n)^{-1}(A) : A \subseteq \{-1,1\}^n\}$

- T - bounded stopping time

$E[S_T] = 0$, $\quad E[S_T^2] = E[T]$

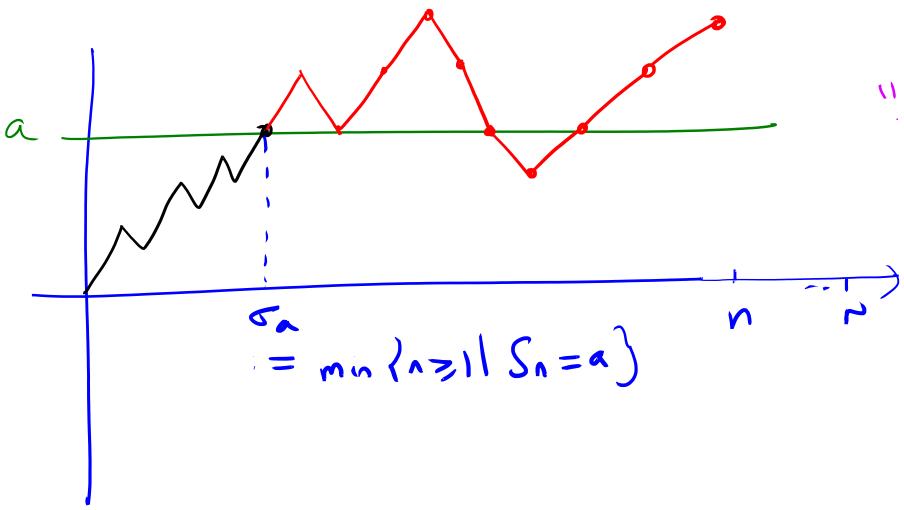
- [Game System] $\quad V_k : \Omega \to \mathbb{R}$ such that $\left.\right\}$ $V_k > 0$, < 0, $= 0$

$\{V_k = c\} \in \mathcal{A}_{k-1}$ $\quad \forall c \in \mathbb{R}$, $k = 1,\dots,N$

$$S_N^V = \sum_{k=1}^{N} V_k X_k$$

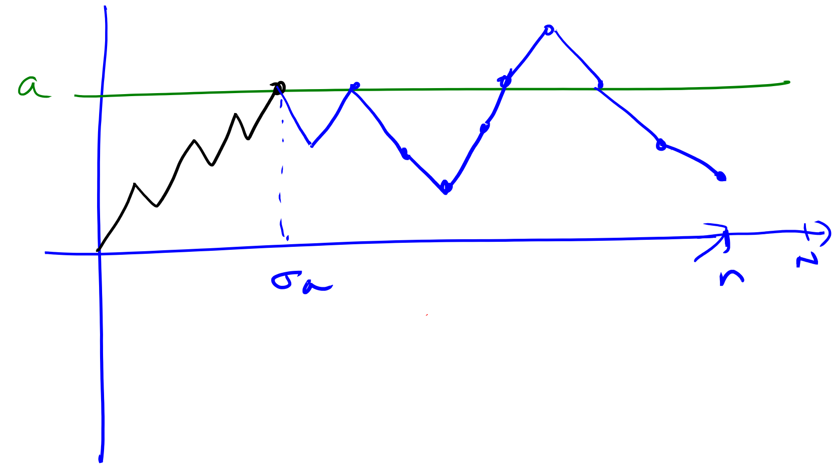
$$E[S_N^V] = 0$$

• Reflection principle

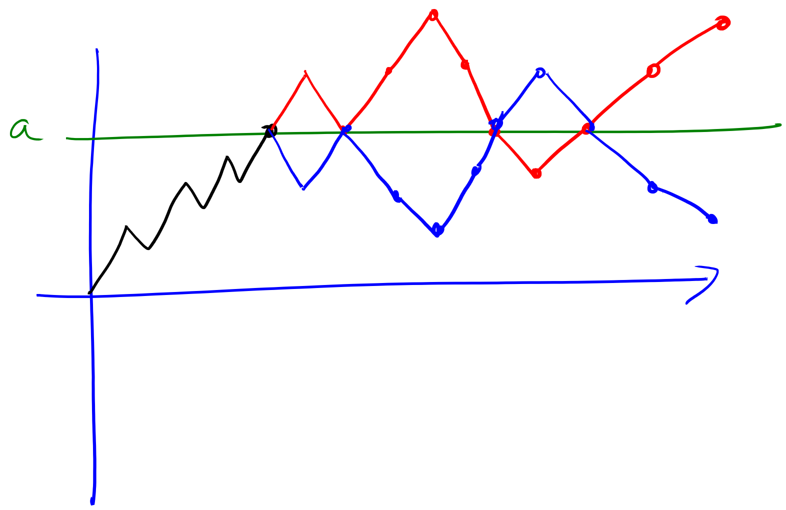


$= \min\{n \geq 1 \mid S_n = a\}$

"reflected" path post hitting a



Reflection principle



Theorem 1.12

• $\mathbb{P}(\sigma_a = n) = \frac{1}{2} \mathbb{P}(S_n = a-1) - \mathbb{P}(S_{n-1} = a+1)$

• $\mathbb{P}(\sigma_a \leq n) = \mathbb{P}(S_n \notin [-a, a-1])$

1.4 Arc Sine law for last time at origin

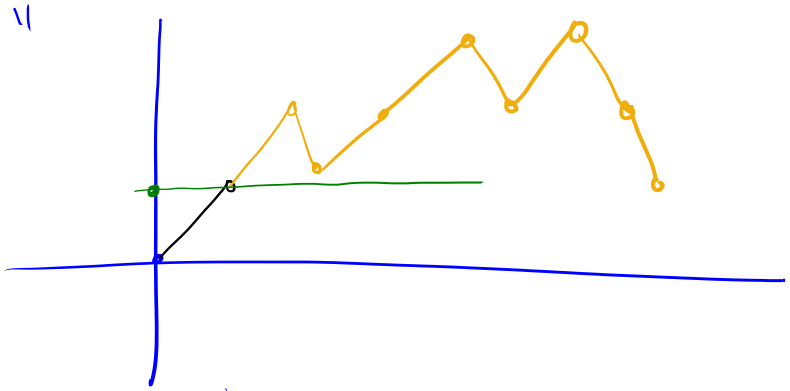
• $L = \max\{0 \leq n \leq 2N : S_n = 0\}$ [Ex: L is not a stopping time]

– What is distribution of L?

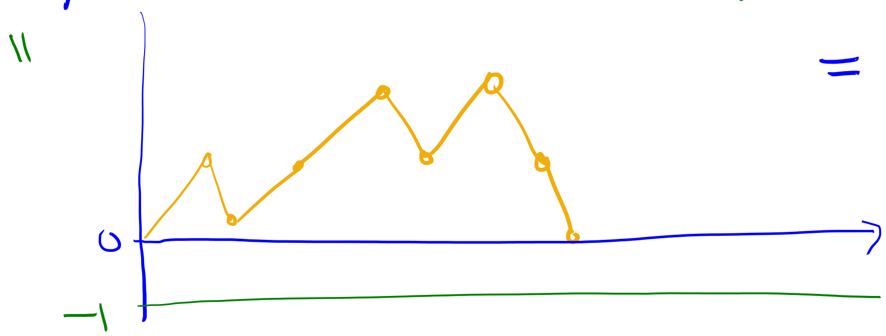
Corollary 1.14: $\mathbb{P}(\sigma_0 > 2n) = \mathbb{P}(S_{2n} = 0)$

Proof :- $\mathbb{P}(\sigma_0 > 2n) = \mathbb{P}(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0)$

[Application of technique in reflection principle] $\overset{(Ex.)}{=} 2\,\mathbb{P}(S_1 > 0, \ldots, S_{2n} > 0)$



$= 2 \cdot \dfrac{\#\text{ of paths start at 1 stay positive for } 2n-1 \text{ steps}}{2^n}$



$= \dfrac{1}{2^{n-1}}$ # of paths start at 0 and stay above −1 for 2n−1 steps

"Reflection principle" $= \dfrac{1}{2^{n-1}}$ # path start at 0 and stay below 1 for 2n−1 steps

$= \mathbb{P}(\sigma_1 > 2n-1)$

$= 1 - \mathbb{P}(\sigma_1 \leq 2n-1) = 1 - \mathbb{P}(S_{2n-1} \notin [-1, 0])$

$\overset{Ex.}{=} \mathbb{P}(S_{2n-1} = -1) = \mathbb{P}(S_{2n} = 0)$ $\square$

Theorem 1.17 : $n \in \mathbb{N} \cup \{0\}$ $n \leq N$

$$\mathbb{P}(L = 2n) = \mathbb{P}(S_{2n} = 0)\, \mathbb{P}(S_{2N-2n} = 0)$$

$$= \frac{1}{2^{2N}} \binom{2n}{n} \binom{2N-2n}{N-n}$$

Proof:- # of paths of length $2N$ with $L = 2n$

$=$ # of paths of length $2n$ with $S_{2n} = 0$ $\times$ # of paths of length $2N-2n$ with $\sigma_0 > 2N-2n$

$$\mathbb{P}(L=2n) \underset{\text{Independent increments}}{=} \mathbb{P}(S_{2n}=0) \times \mathbb{P}(\sigma_0 > 2N-2n)$$

$$= \mathbb{P}(S_{2n}=0) \cdot \mathbb{P}(S_{2N-2n}=0)$$

$= \ldots$ [Use distribution of S_n to complete proof of the result] $\square$

Stirling's formula :- $0 \leq n \leq N$

$$\mathbb{P}(L=2n) = \frac{1}{2^{2N}} \binom{2n}{n} \binom{2N-2n}{N-n} \overset{\text{Ex}}{\simeq} \frac{1}{N} f\left(\frac{n}{N}\right)$$

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}} \qquad x \in (0,1)$$

$\Rightarrow$

$$\mathbb{P}\left(\frac{L}{2N} \leq x\right) = \mathbb{P}(L \leq 2Nx)$$

$$= \sum_{n=0}^{\text{"}n=\lfloor Nx \rfloor\text{"}} \mathbb{P}(L=2n)$$

arc Sine law

- $E[X_1] = 0$
- $\mathbb{P}(S_{2n}=0) \sim \frac{1}{\sqrt{n}}$

$$\overset{\text{Ex}}{\simeq} \sum_{n=0}^{Nx} \frac{1}{N} f\left(\frac{n}{N}\right)$$

$$\approx \int_0^x f(y)\,dy = \int_0^x \frac{dy}{\pi\sqrt{y(1-y)}}$$

$$= \frac{2}{\pi} \arcsin(\sqrt{x}) \qquad \square$$

Towards mean hitting times

Theorem 1.12 $\longrightarrow$

- $$\mathbb{P}(\sigma_a = n) = \frac{1}{2}\left[\mathbb{P}(S_{n-1} = a-1) - \mathbb{P}(S_{n-1} = a+1)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2^{n-1}}\binom{n-1}{\frac{n+a}{2}-1} - \frac{1}{2^{n-1}}\binom{n-1}{\frac{n+a}{2}}\right]$$

(Ex in induction) $\longleftarrow$ $$= \frac{1}{2^n}\,\frac{a}{n}\binom{n}{\frac{n+a}{2}}$$

$$= \frac{a}{n}\,\mathbb{P}(S_n = a) \quad ! \qquad \square$$

Corollary 1.15

$\mathbb{P}(\sigma_0 > 2n) = \mathbb{P}(S_{2n}=0) \underset{\text{Stirling}}{\simeq} \frac{1}{\sqrt{n}} \quad \left(\Rightarrow \sum_{n=0}^{\infty} p_{00}^{n} = \infty \right) \; (+)$

$\Rightarrow \lim_{N\to\infty} \mathbb{P}(\sigma_0 > 2N) = 0$

"Assume we know construction of $N=\infty$ SRW"

$$E(\sigma_0) \overset{!}{=} \sum_{n=0}^{\infty} \mathbb{P}(\sigma_0 > n)$$

(use Corollary 1.14 proof) Ex

$$\overset{\Leftarrow}{=} 2 \sum_{n=0}^{\infty} \mathbb{P}(\sigma_0 > 2n)$$

$$= 2 \sum_{n=0}^{\infty} \mathbb{P}(S_{2n}=0)$$

$$\sim 2 \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}} = \infty \quad - (*)$$

Formal Calculation

(null) (*) Recurrent (+); but you have wait a very long time to return 0.

Corollary 1.10 :- $a \in \mathbb{N}$

$$\mathbb{P}(S_n = a, \sigma_0 > n) = \frac{a}{n} \mathbb{P}(S_n = a)$$

Proof:

$=$ # n steps paths from 0 to a without return 0

$=$ # n step paths from a to 0 without return 0.

(Time reversibility) $=$ # n steps paths from 0 to a with $\sigma_a = n$

$\square$

Extension -

I: Higher dimension

$N \in \mathbb{N}$

$d \geq 1$; $x \in \mathbb{Z}^d$ $\quad |x| = \sqrt{\sum_{j=1}^{d} x_j^2}$

$$\Omega_N = \{ \omega = (\omega_1, \dots, \omega_N) \mid \omega_k \in \mathbb{Z}^d, \ |\omega_k| = 1, \ \forall 1 \leq k \leq N \}$$

$1 \leq k \leq N$, $X_k(\omega) = \omega_k$

$$\Rightarrow \quad S_n(\omega) = \sum_{k=1}^{n} X_k(\omega) \qquad 1 \leq n \leq N$$

$$\begin{pmatrix} S_n^{(1)} \\ \vdots \\ S_n^{(d)} \end{pmatrix}$$

$S_0 = 0$

$S_n^{(j)} \in \mathbb{Z}$

$j = 1, \dots d.$

$A \subseteq \Omega_N$;

$$\mathbb{P}(A) = \frac{|A|}{(2d)^N}$$

II Infinite time Horizon :

$\boxed{N \to \infty}$ $\quad \Omega_N = \{-1, 1\}^N$, $N \in \mathbb{N}$

- $K < M$ $\quad \pi_K : \Omega_M \to \Omega_K$

$$\pi_k(\omega) = (\omega_1, \dots, \omega_k)$$

- Construct as before. $\quad (\Omega_k, \mathcal{A}_k, \mathbb{P}_k)$

"observable till time k"

- [Consistency across "k"] $k < M$, $\tilde{\omega} \in \Omega_k$

$$\mathbb{P}_M \left(\{ \omega \in \Omega_M \mid \Pi_k(\omega) = \tilde{\omega} \} \right)$$

$$= \frac{\# \{ \omega \in \Omega_M \mid \Pi_k(\omega) = \tilde{\omega} \}}{(2d)^M}$$

"only fixing first k-coordinates and allowing M-k to be free"

$$= \frac{(2d)^{M-k}}{(2d)^M}$$

$$= \frac{1}{(2d)^k} = \mathbb{P}_k(\{\tilde{\omega}\})$$

- <u>Kolmogorov Consistency Theorem</u> : [Using above]

$$\Omega_\infty = \{ \omega = (\omega_k)_{k \geq 1} \mid \omega_k \in \mathbb{Z}^d, \ |\omega_k| = 1 \}$$

$$\Pi_k : \Omega_\infty \longrightarrow \Omega_k$$

$$\Pi_k(\omega) = (\omega_1, \ldots, \omega_k)$$

$\rightsquigarrow$ $\mathcal{A}_\infty$ = "suitably define" = "contain $\mathcal{A}_k : k \geq 1$"

$\exists \ \mathbb{P} . \ \mathcal{A}_\infty \to [0,1]$ s.t.

$\tilde{\omega} \in \Omega_k$, $\mathbb{P}\left(\{ \omega \in \Omega_\infty \mid \Pi_k(\omega) = \tilde{\omega} \} \right) = \mathbb{P}_k(\{\tilde{\omega}\})$

$$= \left(\frac{1}{2d}\right)^k$$

$(S_n)_{n \geq 0}$ on $(\Omega_\infty, \mathcal{A}_\infty, \mathbb{P})$ are called

Simple random walk starting at 0

- **Remarks** :-

- Use / understand finite length computation and work with $\mathbb{P}$ for infinite time horizon questions

Results :-

- Strong law large of numbers
- Central limit Theorem
- Large deviation
- Recurrence
- Transience

$$(\Omega_\infty, \mathcal{A}_\infty, \mathbb{P}) \rightsquigarrow (\Omega_N, \mathcal{A}_N, \mathbb{P}_N)$$