Recall :-

$N \in \mathbb{N}$

- $\Omega_N = \{-1,1\}^N$ $\quad \mathcal{F} = \mathcal{P}(\Omega_N)$ $\quad \mathbb{P}: \mathcal{F} \to [0,1]$, $\mathbb{P}(A) = \frac{|A|}{2^N}$ $\quad A \subseteq \Omega_N$

- $k \le N$ $\quad X_k : \Omega_N \to \{-1,1\}$, $X_k(\omega) = \omega_k$; $\quad \underbrace{S_n = \sum_{k=1}^{n} X_k(\omega)}$, $S_0 = 0$ $\quad n \ge 1$ & $n \le N$

finite length random walk.

- $\mathbb{P}(S_n = x) = \mathbb{P}(S_n = -x) = \dfrac{n!}{\frac{n-x}{2}! \cdot \frac{n+x}{2}!}$ $\quad x \in \{-n, -n+2, \dots, n\}$

- Markov chain; independent increments, conditional law

- $A \subseteq \Omega$ - observable by time n if it was a union of basic event $\{\omega \in \Omega \mid \underline{\omega_1 = a_1, \dots \omega_n = a_n}\}$ $\quad a_i \in \{-1,1\}$

$\mathcal{A}_n = \{A \subseteq \Omega \mid A \text{ is observable by time } n\}$ $\quad \phi \in \mathcal{A}_n$

$\longrightarrow$ closed under complements, unions & intersection

Filtration

$$\{\phi, \Omega\} := \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \dots \subseteq \mathcal{A}_n \subseteq \dots \quad \mathcal{A}_N = \mathcal{F}_N$$

- $T : \Omega \to \{0,1,\dots N\} \cup \{\infty\}$ is called a stopping time if $\{T = k\} \in \mathcal{A}_k$ $\quad \forall k = 1, \dots, N$

Theorem 1.8 [No Profit at favorite times]

Let $T : \Omega \to \{0,1,\dots,N\}$ be a stopping time. Then

$$E[S_T] = 0$$

- [Suppress N in notation, ie $\Omega = \Omega_N$ ... and so on.]

Proof :-

$S : \Omega \to \mathbb{Z}$ $\quad T : \Omega \to \{0, 1, \dots, N\}$

$S_T : \Omega \to \mathbb{Z}$ $\quad S_{T(\omega)}(\omega) =: S_T(\omega)$

$S_0 = 0$ $\quad E[S_T] = E\left[\sum_{k=1}^{N} S_k \mathbb{1}_{(T=k)}\right]$, where $\mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \text{otherwise} \end{cases}$

Now, $\sum_{k=1}^{N} S_k \mathbb{1}_{(T=k)} = \sum_{k=1}^{N} S_k \left[\mathbb{1}_{T \geq k} - \mathbb{1}_{T \geq k+1}\right]$

$$= \sum_{k=1}^{N} (S_k - S_{k-1}) \mathbb{1}(T \geq k)$$

$$= \sum_{k=1}^{N} X_k \mathbb{1}_{T \geq k}$$

$$\therefore \quad E[S_T] = \sum_{k=1}^{N} E[X_k \mathbb{1}_{T \geq k}]$$

$1 \leq k \leq N$, $\quad X_k \mathbb{1}_{T \geq k} = \begin{cases} 1 & X_k = 1 \ \& \ T \geq k \\ -1 & X_k = -1 \ \& \ T \geq k \\ 0 & \text{otherwise} \end{cases}$

$$\therefore \quad E[X_k \mathbb{1}_{T \geq k}] = \mathbb{P}(X_k = 1, T \geq k) - \mathbb{P}(X_k = -1, T \geq k)$$

- $\{T \geq k\}^c = \bigcup_{\ell=1}^{k-1} \{T = \ell\} \in \mathcal{A}_{k-1} \implies \{T \geq k\} \in \mathcal{A}_{k-1}$

"Stopping time is greater than or equal to k should be determined by $X_1, \dots, X_{k-1}$"

- X_k is independent of $X_1, \dots, X_{k-1}$

$\implies$ $X_k = 1$ is $\perp$ of $T \geq k$

$X_k = -1$ is $\perp$ of $T \geq k$

$\therefore$ $\mathbb{P}(X_k = 1,\ T \geq k) = \mathbb{P}(X_k = 1)\ \mathbb{P}(T \geq k)$
$\mathbb{P}(X_k = -1,\ T \geq k) = \mathbb{P}(X_k = -1)\ \mathbb{P}(T \geq k)$ $\Big\} = \frac{1}{2}\mathbb{P}(T \geq k)$

$(+)$ $1 \leq k \leq \infty$

$$\therefore\ E[X_k \mathbb{1}_{T \geq k}] = \mathbb{P}(X_k = 1, T \geq k) - \mathbb{P}(X_k = -1, T \geq k)$$
$$= \frac{1}{2}\mathbb{P}(T \geq k) - \frac{1}{2}\mathbb{P}(T \geq k) = 0$$

$$\Rightarrow \quad E[S_T] = \sum_{k=1}^{N} E[X_k \mathbb{1}_{T \geq k}] = 0$$

$\square$

Question: What is $\mathrm{Var}[S_T]$? T - stopping time.

Definition 1.8.1: A game system is a sequence of $\mathbb{R}$ valued random variables $\{V_k\}_{k=1}^{N}$

- $V_k : \Omega \to \mathbb{R}$ such that

$\{V_k = c\} \in \underline{\mathcal{A}_{k-1}}$ $\forall\, c \in \mathbb{R}$, $k = 1, \dots N$

" $V_k \equiv$ amount we bet at start of k^{th} round"

$\Big[\ V_k > 0,\ \ V_k = 0,\ \ V_k < 0\ \Big]$ ✓

"Result" at k^{th} round : $V_k X_k$

(Total gain of the game). $S_N^V = \sum_{k=1}^{N} V_k X_k$.

Theorem 1.9 :- [Profitability is impossible!]

For any game system $V_1, V_2, \dots, V_N$ the expected total gain of the game vanishes.

$$E[S_N^V] = 0$$

Proof :-

$$E[S_N^V] = E\left[\sum_{k=1}^{N} V_k X_k\right]$$

$$= \sum_{k=1}^{N} E[V_k X_k]$$

Range of $V_k \equiv \{ c_i^k : 1 \le i \le m_k \}$ & $V_k = \sum_{i=1}^{m_k} c_i^k \mathbb{1}(V = c_i^k)$

$$= \sum_{k=1}^{N} \sum_{i=1}^{m_k} c_i^k \, E[X_k \underbrace{\mathbb{1}(V_k = c_i^k)}]$$

$X_k \perp A_{k-1}$

independent

"as in proof of"
$E[S_T] = 0$
at (+)

$$= \sum_{k=1}^{N} \sum_{i=1}^{m_k} c_i^k \cdot 0 \qquad \square$$

$$= 0$$

Theorem 1.10 $T : \Omega \to \{0, 1, \dots, N\}$

$$Var[S_T] = E[T]$$

Proof :-

$$\text{Var}[S_T] = E[S_T^2] - (E[S_T])^2$$

$$\underset{\text{Theorem 1.8}}{=} E[S_T^2] - 0$$

$$S_T^2 = \sum_{k=1}^{N} S_k^2 \, 1_{(T=k)} = \sum_{k=1}^{N} (S_k^2 - S_{k-1}^2) \, 1_{(T \geq k)} \quad \text{— (++)}$$

Now,

$$S_k^2 = (S_{k-1} + X_k)^2 = S_{k-1}^2 + X_k^2 + 2X_k S_{k-1}$$

$$= S_{k-1}^2 + 1 + 2X_k S_{k-1}$$

$$\Rightarrow \quad S_k^2 - S_{k-1}^2 = 1 + 2 X_k S_{k-1} \quad \text{— (*)}$$

$\therefore$ (*) into (++) we have

$$S_T^2 = \sum_{k=1}^{N} (S_k^2 - S_{k-1}^2) \, 1_{(T \geq k)}$$

$$= \sum_{k=1}^{N} (1 + 2 X_k S_{k-1}) \, 1_{T \geq k}$$

$$= \sum_{k=1}^{N} 1 \, 1_{T \geq k} + 2 \sum_{k=1}^{N} X_k \underbrace{S_{k-1} \, 1_{T \geq k}}_{\text{"}V_k\text{"}}$$

$$\therefore \quad S_T^2 = T + 2 \sum_{k=1}^{N} X_k V_k$$

where $V_k = S_{k-1} \, 1_{T \geq k}$.

$\therefore \quad E[S_T^2] = E[T] + 2\, E[S_N^V]$

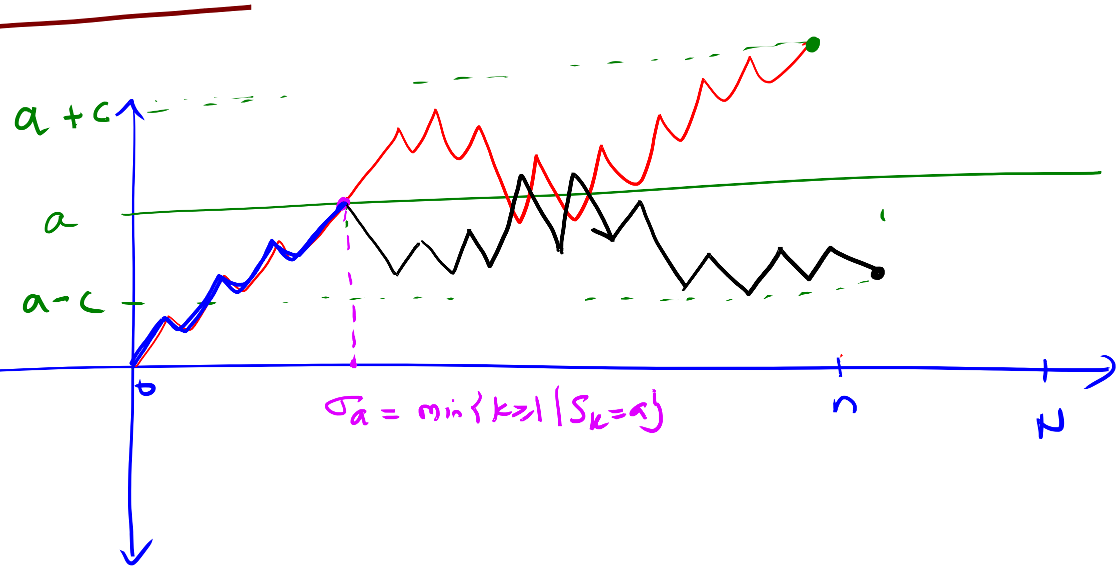
where $V_k = \underbrace{S_{k-1}\, 1_{T \geq k}}$ — A_{k-1} determined random variables

$\Rightarrow \{V_k = c\} \in A_{k-1}$

$\therefore$ Theorem 1.9 $\Rightarrow$

$$E[S_T^2] = E[T] \qquad \square$$

1.3 Reflection Principle :-



each path after σ_a
$\equiv$
for every red that is at a+c "reflection" there is a black that is at a-c

'Black $\equiv$ reflection across $y=a$ of Red

$c > 0$

$\Rightarrow$

$1 \leq n \leq N$

$$|\{S_n = a+c\}| = |\{\sigma_a \leq n,\ S_n = a+c\}|$$

has to happen as $S_n = a+c$

$$= |\{\sigma_a \leq n,\ S_n = a-c\}|$$

$\boxed{\text{Reflection Principle}}$

Lemma 1.11:- $a, c \in \mathbb{N}$

$$\mathbb{P}(S_n = a-c, \ \sigma_a \le n) = \mathbb{P}(S_n = a+c)$$

Theorem 1.12: ⓐ $\mathbb{P}(\sigma_a \le n) = \mathbb{P}(S_n \notin [-a, a-1])$

(HW) ⓑ $\mathbb{P}(\sigma_a = n) = \frac{1}{2}\left[\mathbb{P}(S_{n-1} = a-1) - \mathbb{P}(S_{n-1} = a+1)\right]$

Proof:- ⓐ

$$\mathbb{P}(\sigma_a \le n) = \sum_{b \in \mathbb{Z}} \mathbb{P}(\sigma_a \le n, \ S_n = b)$$

$$= \sum_{\substack{b \in \mathbb{Z} \\ b \ge a}} \mathbb{P}(\sigma_a \le n, \ S_n = b) + \sum_{\substack{b \in \mathbb{Z} \\ b < a}} \mathbb{P}(\sigma_a \le n, \ S_n = b)$$

"$b = a-c$ for $c > 0$"

Reflection

lemma 1.11 with $b = a-c$

$$= \sum_{\substack{b \in \mathbb{Z} \\ b \ge a}} \mathbb{P}(S_n = b) + \sum_{\substack{b \in \mathbb{Z} \\ b < a}} \mathbb{P}(S_n = 2a-b)$$

$$= \mathbb{P}(S_n \ge a) + \mathbb{P}(S_n > a)$$

Symmetry

$$= \mathbb{P}(S_n \ge a) + \mathbb{P}(S_n < -a)$$

$$= \mathbb{P}(S_n \notin [-a, a-1])$$

$\square$

1.4 Arc Sine law :

$$L = \max \{ 0 \le n \le 2N : S_n = 0 \}$$

"last visit to the origin"

– NOT a stopping time

Theorem 1.17 : $n \in \mathbb{N} \cup \{0\}$ $\quad n \le N$

$$\mathbb{P}(L = 2n) = \mathbb{P}(S_{2n} = 0)\,\mathbb{P}(S_{2N-2n} = 0) = 2^{-2N} \binom{2n}{n} \binom{2N-2n}{N-n}$$

Proof – next class.

Arc-Sine law :

Stirling's formula

$$\mathbb{P}(L = 2n) \approx \frac{1}{\pi \sqrt{n(N-n)}}$$

$$\mathbb{P}\left(\frac{L}{2N} \le x \right) = \sum_{n=0,\ \frac{n}{N} \le x} \frac{1}{\pi \sqrt{n(N-n)}}$$

.... $N \to \infty$? [Next class]