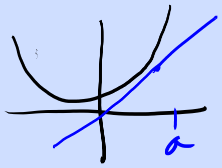
Recall :- **Kolmogorov's inequalities**



- $h : \mathbb{R} \to \mathbb{R}$ st $\forall a \in \mathbb{R}$ $\exists c \in \mathbb{R}$ such that
$$h(x) \geq h(a) + c(x-a) \quad \forall x \in \mathbb{R}$$
} Convex

Jensen's inequality : $E\, h(X) \geq h(E(X))$ when h is convex

- $\{Z_n\}_{n \geq 1}$ was a martingale then
$$E[h(Z_n) \mid Z_{n-1}, \ldots Z_1] \geq h(Z_{n-1})$$

$\{Z_n\}_{n\geq 1}$ — sub-martingale — $E(|Z_n|) < \infty$ $\forall n \geq 1$
$$E[Z_n \mid Z_{n-1}, \ldots, Z_1] \geq Z_{n-1} \quad \forall n \geq 2$$

Theorem 3.8 [Kolmogorov sub-martingale Inequality] $\{Z_n\}_{n\geq 1}$ be a non-negative sub-martingale. Then for any positive integer m and any $a > 0$

$$\mathbb{P}\left(\max_{1 \leq i \leq m} Z_i \geq a \right) \leq \frac{E[Z_m]}{a} \quad — \quad (1)$$

Observation:
- Compare (1) with Markov inequality
- any martingale is also a sub-martingale

Proof :- let $m \in \mathbb{N}$ be given

$J = \min\{k \geq 1 \mid Z_k \geq a\} \wedge m$

is a bounded stopping time

- Earlier HW we have seen that J is a stopping time.

Note :- $Z_n < a \ \forall \ n < m \implies J = m$.

So $Z_J \geq a$ iff $Z_n \geq a$ for some $n \leq m$.

$$\implies \mathbb{P}\left(\max_{1 \leq n \leq m} Z_n \geq a\right) = \mathbb{P}(Z_J \geq a)$$

$(Z_i \geq 0)$ ..... (Markov inequality)

$$\leq \frac{E[Z_J]}{a}$$

($Z_J = Z_{J \wedge m}$ as J is bounded by m)

$$= \frac{E[Z_m^J]}{a} \quad — \ (+)$$

Now, $Z_m^J = Z_{J\wedge m} = \sum_{k=1}^{m} Z_k 1_{J=k} + Z_m 1_{J>m}$

$$\implies E[Z_m^J] = \sum_{k=1}^{m} E[Z_k 1_{J=k}] + E[Z_m 1_{J>m}]$$

Z_m - sub-martingale

$(E[Z_m \mid A_k] \geq Z_k)$, $1 \leq k \leq m$

$$\leq \sum_{k=1}^{m} E\left[E[Z_m \mid A_k] 1_{J=k}\right] + E[Z_m 1_{J>m}]$$

Conditional Expectation and (predictable)

ie $E[XY \mid A_y] = Y E[X \mid A_y]$

$$= \sum_{k=1}^{m} E\left[E[Z_m 1_{J=k} \mid A_k]\right] + E[Z_m 1_{J>m}]$$

$E[E[X|A_X]] = E[X]$ TP

$$\sum_{k=1}^{m} E[Z_m 1_{J=k}] + E[Z_m 1_{J>m}]$$

$$= E\left[Z_m \left(\sum_{k=1}^{m} 1_{J=k} + 1_{J>m}\right)\right]$$

$$= E[Z_m] \quad — \quad (*)$$

Using $(*)$ and $(\dagger)$ we have

$$\mathbb{P}\left(\max_{1\le i\le m} Z_i \ge a\right) \le \frac{E[Z_m]}{a} \quad \forall m\ge 1$$

$\square$

Recall: Corollary 3.9: let $\{Z_m\}_{m\ge 1}$ be a martingale with $E[Z_m^2] < \infty$ $\forall m\ge 1$ then

$$\mathbb{P}\left(\max_{1\le i\le m} |Z_i| \ge b\right) \le \frac{E[Z_m^2]}{b^2} \quad \forall m\ge 2 \quad \forall b>0.$$

Proof: Apply Theorem 3.8 to Z_i^2 and $a=b^2$ $\square$

Corollary 3.10 :- let $\{Z_n : n\ge 1\}$ be a non-negative martingale

$$\mathbb{P}\left(\sup_{n\ge 1} Z_n \ge a\right) \le \frac{E[Z_1]}{a} \quad \forall a>0$$

Proof: $\{Z_m\}_{m\geq 1}$ is a martingale (non-negative)

Theorem 3.8 $\Rightarrow$ $\mathbb{P}\left(\max_{1\leq i\leq m} Z_i \geq a\right) \leq \dfrac{E[Z_m]}{a}$

(Earlier Corollary & lemma 3.2 :- $E[Z_m] = E[Z_1]$ $\forall m\geq 1$) $= \dfrac{E[Z_1]}{a}$ — $(*)$

$A_m = \left\{\max_{1\leq i\leq m} Z_i \geq a\right\} \Rightarrow A_m \subseteq A_{m+1}$ $\forall m\geq 1$

$A_m \uparrow A = \bigcup_{i\geq 1}\{Z_i \geq a\}$ [key point] $(**)$

let $\varepsilon_0 > 0$ be given. and $0 < \varepsilon_0 < a$.

$\mathbb{P}\left(\sup_{n\geq 1} Z_n \geq a\right) = \mathbb{P}\left(\bigcap_{\varepsilon>0} \bigcup_{i=1}^{\infty}\{Z_i \geq a-\varepsilon\}\right)$

$\leq \mathbb{P}\left(\bigcup_{i=1}^{\infty}\{Z_i \geq a-\varepsilon_0\}\right)$ — $(***)$

Take (I): $a-\varepsilon_0$ in $(*)$ to get

$\underbrace{\mathbb{P}\left(\max_{1\leq i\leq m} Z_i > a-\varepsilon_0\right)}$ $\leq \dfrac{E[Z_1]}{a-\varepsilon_0}$

(II):

(Take limit as $n\to\infty$ in I) $\mathbb{P}\left(\bigcup_{i\geq 1} Z_i > a-\varepsilon_0\right) \leq \dfrac{E[Z_1]}{a-\varepsilon_0}$

and the $A_n \uparrow A$

$\Rightarrow \mathbb{P}(A_n) \uparrow \mathbb{P}(A)$

III Use (xxx) to get

$$\mathbb{P}\left(\sup_{n\geq 1} Z_n \geq a\right) \leq \frac{E[Z_1]}{a-\varepsilon_0}$$

IV $\varepsilon_0 > 0$ was arbitrary in $0 < \varepsilon_0 < a$

$$\Rightarrow \mathbb{P}\left(\sup_{n\geq 1} Z_n \geq a\right) \leq \frac{E[Z_1]}{a} \quad \square$$

Theorem 3.11 (Martingale Convergence Theorem) :-

let $\{Z_n\}_{n\geq 1}$ be a a martingale and assume that $\exists M>0$ such that $E[Z_n^2] \leq M \ \forall n \geq 1$

Then there exist a r.v Z such that

$Z_n \longrightarrow Z$ as $n \to \infty$ with probability 1.

(ie $\mathbb{P}\left(\lim_{n\to\infty} Z_n = Z\right) = 1$ or $\mathbb{P}(Z_n \to Z \text{ as } n\to\infty) = 1$)

Proof :- $\{Z_n^2\}_{n\geq 1}$ is a sub-martingale (non-negative)

$$\Rightarrow E[Z_n^2 \mid Z_{n-1}^2, \ldots Z_1^2] \geq Z_{n-1}^2 \quad \forall n \geq 2$$

(TP) $$E[Z_n^2 \mid Z_i^2, \ldots Z_1^2] \geq Z_i^2 \quad \forall n \geq i \geq 1 \quad (*)$$

(Above, At $i=1$ $E[Z_n^2 \mid Z_1^2] \geq Z_1^2$)

Taking Expectation in (*) $\quad E[Z_n^2] \geq E[Z_i^2] \quad \forall n \geq i \geq 1$

In particular $\quad E[Z_n^2] \geq E[Z_{n-1}^2] \quad \forall n \geq 2. \quad$ — (+)

($E[Z_n^2]$ is increasing sequence)

(+) and assumption that : $\quad E[Z_n^2] \leq M \quad \forall n \geq 1$

$\exists\, M > 0$

(Bounded above)

$\Rightarrow \quad E[Z_n^2] \longrightarrow \alpha$ (say) as $n \to \infty$. — (I)

Next, let $k \geq 1$ and $Y_n = Z_{k+n} - Z_k \quad \forall n \geq 1$

Claim II :- $\{Y_n\}_{n \geq 1}$ is a mean 0, martingale.

Proof :- In HW 7, Book-keeping Ex D

$$E[Y_n^2] = E[(Z_{k+n} - Z_k)^2]$$

$$= E[Z_{k+n}^2] + E[Z_k^2] - 2E[Z_{k+n} Z_k]$$

(TP) $\quad = E[Z_{k+n}^2] + E[Z_k^2] - 2E\big[E[Z_{k+n} Z_k \mid A_k]\big]$

Conditional Expectation & Predictable $\quad = E[Z_{k+n}^2] + E[Z_k^2] - 2E\big[Z_k\, E[Z_{k+n} \mid A_k]\big]$

$\{Z_n\}_{n \geq 1}$ is martingale $\quad = E[Z_{k+n}^2] + E[Z_k^2] - 2E[Z_k^2]$

$$= E[Z_{k+n}^2] - E[Z_k^2] < \infty \quad \text{— (II)}$$

Apply Corollary 3.9 $\Rightarrow$ $\forall b>0$ & $m \geq 1$

$$\mathbb{P}\left(\max_{1\leq i\leq m} |Y_i| \geq b\right) \leq \frac{E[Y_m^2]}{b^2}$$

$$\Rightarrow \quad \mathbb{P}\left(\max_{1\leq i\leq m} |Z_{i+k} - Z_k| \geq b\right) \leq \frac{E[Z_{k+m}^2] - E[Z_k^2]}{b^2}$$

let $m \to \infty$ on both sides to get

$$\forall b>0 \quad \mathbb{P}\left(\bigcup_{i\geq 1}\{|Z_{i+k} - Z_k| \geq b\}\right) \overset{\text{I}}{\leq} \frac{\alpha - E[Z_k^2]}{b^2}$$

- A similar argument as before in Proof of Corollary 3.10 will provide

$$\forall b>0, \quad 0 \leq \mathbb{P}\left(\sup_{i\geq 1} |Z_{i+k} - Z_k| \geq b\right) \leq \frac{\alpha - E[Z_k^2]}{b^2}$$

Take $k \to \infty$ on all terms

$$\forall b>0 \quad \lim_{k\to\infty} \mathbb{P}\left(\sup_{i\geq 1} |Z_{i+k} - Z_k| \geq b\right) = 0$$

III above implies $\{Z_k\}_{k\geq 1}$ is a Cauchy sequence with probability 1. [To show] [Using Borel – Cantelli]

$\Rightarrow \quad \exists Z \quad \text{st} \quad \mathbb{P}(Z_n \to Z \text{ as } n \to \infty) = 1$ $\square$

Q2: HW 7 - Book-keeping Exercise: $\{Z_n\}_{n\geq 1}$ to be a martingale & $Y_n = Z_{k+n} - Z_k$ for some $k \geq 1$, $\forall n \geq 1$

Claim: $\{Y_n\}_{n\geq 1}$ is a mean 0 martingale

Proof: $\forall n \geq 1$

$$E|Y_n| \leq \underbrace{E|Z_{k+n}| + E|Z_k|}_{<\infty} \quad \forall n \geq 1$$

$$E[Y_n] = E[Z_{k+n}] - E[Z_k] \overset{\text{martingale}}{=} E[Z_1] - E[Z_1] = 0$$

let A_n - be the observable events by time n w.r.t. $\{Z_m\}_{m\geq 1}$

B_n - be the observable event by time n wrt $\{Y_m\}_{m\geq 1}$

(C - observable by time n wrt Y)

$\Updownarrow$

$1_C = f(Y_1, \ldots, Y_n)$ for some f

$\Leftarrow$

$1_C = g(Z_k, \ldots Z_{n+k})$ for some g

$\Uparrow$

(C is observable by time $n+k$ wrt Z)

So $\mathcal{A}_{k+n} \supseteq \mathcal{B}_n$

To show: $E(Y_n \mid \mathcal{B}_{n-1}) = Y_{n-1}$, it is

enough to show

$$E(Y_n \mid \mathcal{A}_{k+n-1}) = Y_{n-1}$$

$$\iff E\left(Z_{k+n} - Z_k \mid \mathcal{A}_{k+n-1}\right) = Y_{n-1}$$

$$\iff \underbrace{E(Z_{k+n} \mid \mathcal{A}_{k+n-1})}_{\{Z_k\} \text{ - martingale}} - \underbrace{E[Z_k \mid \mathcal{A}_{k+n-1})}_{\substack{\text{Conditional expectation} \\ \leftarrow \text{predictable}}} = Y_{n-1}$$

$$\iff Z_{k+n-1} - Z_k = Y_{n-1}$$

which is true $\square$

Question 1 <u>HW 7 - Book-keeping exercise</u>:

$T : \Omega \to \mathbb{N}$ be a stopping time w.r.t. martingale $\{M_n\}_{n \geq 1}$

- discrete r.v.'s -

- $\mathbb{Z}$-valued -

$$E[M_1 \mathbb{1}_{T>N}] \longrightarrow 0 \quad \text{as } N \to \infty.$$

Q1: $E|M_1| < \infty$, T is a r.v such that

(M_1 - Discrete random variable valued in $\mathbb{Z}$)

$\mathbb{P}(T<\infty) = 1 \Rightarrow E[M_1 \mathbb{1}_{T>N}] \to 0$ as $N \to \infty$

Proof: $\lim_{n\to\infty} E[|M_1| \mathbb{1}_{T>N}] \overset{?}{=} \sum_{k=1}^{\infty} \underbrace{\lim_{N\to\infty} \mathbb{P}(|M_1| = k, T>N)}_{= 0} k$

let $\varepsilon > 0$ be given.

Given :- $E|M_1| = \sum_{k=1}^{\infty} k \, \mathbb{P}(|M_1| = k) < \infty$

Ex: $\exists N_0 : \sum_{k=N_0+1}^{\infty} k \, \mathbb{P}(|M_1| = k) < \varepsilon.$

$$\therefore E[|M_1| \mathbb{1}_{T>N}] \overset{(Ex)}{=} \sum_{k=1}^{N_0} k \, \mathbb{P}(|M_1| = k, T>N) + \underbrace{\sum_{k=N_0+1}^{\infty} k \mathbb{P}(|M_1| = k, T>N)}_{\wedge \, (Ex)}$$

$$\leq \left(\sum_{k=1}^{N_0} k\right) \mathbb{P}(T>N) + \sum_{k=N_0+1}^{\infty} k \, \mathbb{P}(|M_1| = k)$$

$$\Rightarrow 0 < E[|M_1| \mathbb{1}_{T>N}] \leq \frac{N_0(N_0+1)}{2} \mathbb{P}(T>N) + \varepsilon$$

Ex: $\exists N_1$ st $\mathbb{P}(T>N) < \dfrac{\varepsilon}{2(N_0)(N_0+1)}$ $\forall N > N_1$

$\exists N_1$ st $\forall N \geq N_1$

$$0 < E(|M_1| \mathbb{1}(T > N)) < 2\varepsilon.$$

As $\left| E[M_1 \mathbb{1}_{T>N}] \right| \leq E[|M_1| \mathbb{1}_{T>N}]$

$\Rightarrow \exists N_1$ st $\forall N \geq N_1$

$$\left| E[M_1 \mathbb{1}_{T>N}] \right| < 2\varepsilon$$

$$\Rightarrow \lim_{N\to\infty} E[M_1 \mathbb{1}_{T>N}] \to 0 \text{ as } N \to \infty \qquad \square$$