Recall :-

- Simple random walk on $\mathbb{Z}^d$.

$$X_n = X_0 + \sum_{i=1}^{n} \xi_i \qquad \xi_i \equiv \text{i.i.d random variables}$$

- $E[\xi] < \infty$

Finite length simple Random walk on $\mathbb{Z}$

$N \in \mathbb{N} \qquad \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$

- $\Omega_N = \{ \omega = (\omega_1, \omega_2, \ldots \omega_N) \in \{-1,1\}^N \}$
- $1 \leq k \leq N \qquad X_k : \Omega_N \to \{-1,1\}$ be given by
  $X_k(\omega) = \omega_k$
- $S_0 = 0, \quad 1 \leq n \leq N \qquad S_n = \sum_{k=1}^{n} X_k$

Let • $\mathcal{F}_N = \mathcal{P}(\Omega_N)$ & • $\mathbb{P}^N : \mathcal{F}_N \to [0,1]$ be given by

[Uniform distribution] $\quad \mathbb{P}(A) \equiv \mathbb{P}^N(A) := \dfrac{|A|}{2^N} \qquad \forall A \subseteq \Omega_N.$

Ex:- - $\mathbb{P}$ - satisfies Axioms of Probability

- $\mathbb{P}(X_k = 1) = \frac{1}{2} = \mathbb{P}(X_k = -1) \quad \forall 1 \leq k \leq N$

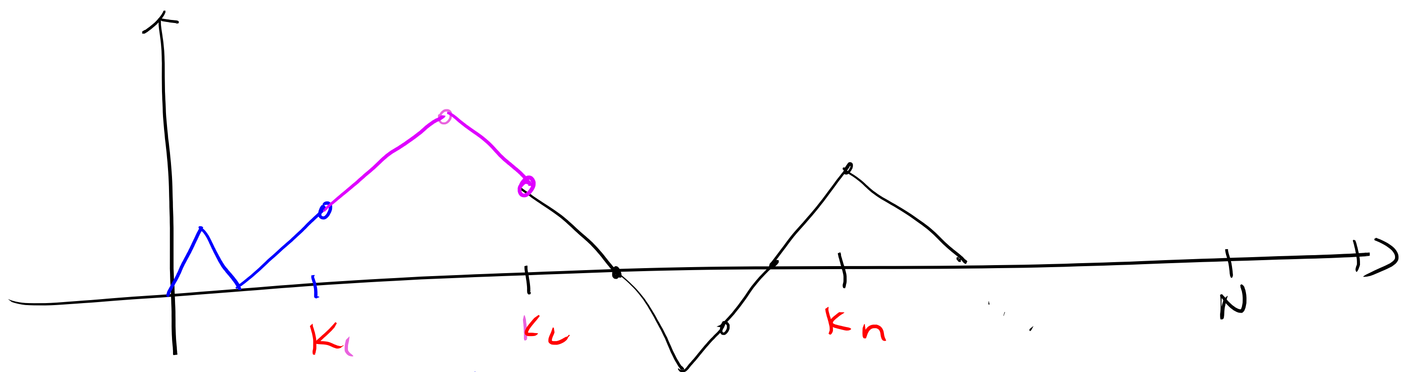
Definition 1.1 :- The sequence of random variables $(S_n)_{n=1}^{N}$ on $(\Omega_N, \mathcal{F}_N, \mathbb{P}^N)$ is called a simple random walk of length N starting at 0.

From now on :- { Suppress N in the notation
view N - some large number

Observation 1.1 :- [Proofs will be left as Exercises]

(a) $\{X_k\}_{k=1}^{N}$ are i.i.d with $\mathbb{P}(X_1=1) = P(X_1=-1) = \frac{1}{2}$.

(b) $1 \le k_1 < k_2 \cdots < k_n \cdots \le N$



- $S_{k_1} - S_0 = \sum_{i=1}^{k_1} X_i$ is independent of $S_{k_2} - S_{k_1} = \sum_{i=k_1+1}^{k_2} X_i$

$\{ S_{k_m} - S_{k_{m-1}} : 1 \le m \le n \}$ are independent

— independent increments

(c) $1 \le k < m \le N$ [Stationary increments]

$$\mathbb{P}(S_m - S_k = a) = \mathbb{P}(S_{m-k} = a) \quad \forall a \in \mathbb{Z}.$$

(d) [Markov Property] $\alpha_i \in \mathbb{Z}$, $1 \le i \le n$, $0 \le n \le N$

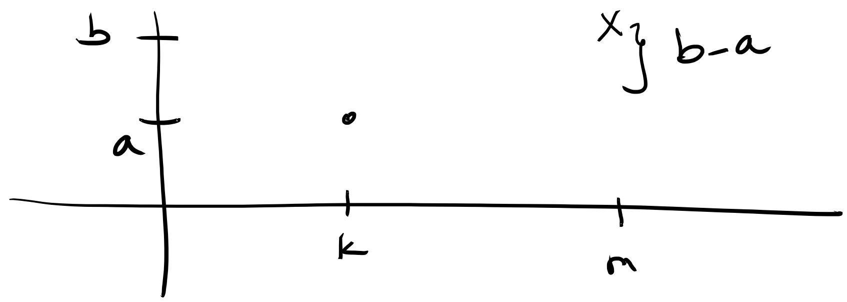
$$\mathbb{P}(S_n = \alpha_n \mid S_{n-1} = \alpha_{n-1}, S_{n-2} = \alpha_{n-2}, \ldots S_1 = \alpha_1, S_0 = 0)$$

$$= \mathbb{P}(S_n = \alpha_n \mid S_{n-1} = \alpha_{n-1})$$

[modulo definition of Conditional Probability]

(e) [Conditional law] $\quad 0 < k < m \leq N$

$$\mathbb{P}(S_m = b \mid S_k = a) = \mathbb{P}(S_{m-k} = b-a)$$



(f)

$$\forall\ 1 \leq k \leq N \begin{cases} - \ E[X_k] = 1 \cdot \frac{1}{2} + (-1)\frac{1}{2} = 0 \ ; \quad E[X_k^2] = 1 \\ - \ E[X_k X_\ell] = 0 \qquad 1 \leq k \neq \ell \leq N \end{cases}$$

$$1 \leq n \leq N \begin{cases} E[S_n] = 0 \ , \quad Var[S_n] = \sum_{i=1}^{n} Var[X_i] = n \\ E[S_n^2] = n \end{cases}$$

[mean & variance]

(g) [Distribution]

$$\mathbb{P}(S_n = 2j-n) = \frac{\left| \left\{ \omega \in \Omega \ \middle| \ \begin{matrix} j \text{ of } \omega_i = 1 \\ n-j \text{ of } \omega_i = -1 \\ 1 \leq i \leq n \end{matrix} \right\} \right|}{2^N} = \frac{\binom{n}{j} 2^{N-n}}{2^N} = \binom{n}{j} \frac{1}{2^n}$$

$\omega = (\omega_1, \ldots, \omega_N)$

$\underline{0 \leq j \leq n}$

$S_n = 2j - n \iff j = \#$ of steps of size $1 \implies n - j = \#$ of steps of size -1

Rewrite:-

$x \in \{-n, -n+2, \ldots, \ , n-2, n\}$

$$\mathbb{P}(S_n = x) = \binom{n}{\frac{n+x}{2}} \left(\frac{1}{2}\right)^n .$$

[Symmetry] around 0 $\mathbb{P}(S_n = x) = \dfrac{n!}{\left(\frac{n-x}{2}\right)!\left(\frac{n+x}{2}\right)!}\dfrac{1}{2^n} = \mathbb{P}(S_n = -x)$

(h) [Mode] - maximal weight of the distribution of S_n

(Ex.)

$$\mathbb{P}(S_{2n} = 0) = \mathbb{P}(S_{2n-1} = 1) = \binom{2n}{n}\frac{1}{2^{2n}}$$

→ (Ex) • $\mathbb{P}(S_{2n} = 0) \sim \dfrac{1}{\sqrt{n}}$

(Ex) • $a, b \in \mathbb{Z}$ $\mathbb{P}(a \leq S_N \leq b) \to 0$ as $N \to \infty$

(g) & (h)

Remark: - $N = \infty$ case involves some subtle-ties (?) involving construction of event space & Probability.

- outline in next tw but observations (a) - (h) hold as well

1.2 Stopping times

- Motivation - [Gambler's Ruin] ; ie S_n = Capital of player one in a 2-player game.

J.M Keynes :- " millionaires should always gamble, poor should never do so"

Question :- Can you stop the game at a favourite moment so as to have a positive expected gain?

Rules :-
- Fair game
- no insider trading
- no knowledge about the future

Favourite moment

$\Downarrow$

what are your observables that you can decide from?

Definition 1.5 :-

An event $A \subseteq \Omega$ is observable until time n when it can be written as a union of basic events of the form

$$\{\omega \in \Omega \mid \omega_1 = o_1, \dots \omega_n = o_n\} \quad \text{with } o_i \in \{-1, 1\}, \ 1 \le i \le n$$

$$\mathcal{A}_n = \{A \subseteq \Omega \mid A \text{ is observable by time } n\}$$

- include ϕ

$\{\mathcal{A}_n\}_{n=0}^{N}$ is called a Filtration

Note :-

$$\{\phi, \Omega\} = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots \subseteq \mathcal{A}_i \subseteq \quad \subseteq \mathcal{A}_N = \mathcal{P}(\Omega)$$

- $\phi, \Omega \in \mathcal{A}_i$
- $A \in \mathcal{A}_i \Rightarrow A^c \in \mathcal{A}_i$
- $\{E_k\}_{k=1}^{\infty} \in \mathcal{A}_i \Rightarrow \bigcup_{k=1}^{\infty} E_k \in \mathcal{A}_i$

Definition 1.7 :- A function $T: \Omega \to \{0, 1, \ldots N\} \cup \{\infty\}$ is called a stopping time if

$$\{T = n\} = \{\omega \in \Omega \mid T(\omega) = n\} \in \mathcal{A}_n \quad \forall\, n = 0, 1, 2, \ldots N$$

Note :- $\{T \le n\} = \bigcup_{k=1}^{n} \{T = k\} \in \mathcal{A}_n. \quad \forall\, n = 0, 1, 2, \ldots, N$

Convention: $\min(\phi) = \infty$

Example 1.7 • $\sigma_a = \min\{n \in \mathbb{N} \mid S_n = a\}$

$1 \le k \le N$

$$\{\sigma_a = k\} = \{\omega \in \Omega \mid S_0 = 0,\ S_1 \ne a, \ldots, \ldots, S_{k-1} \ne a, S_k = a\}$$

$$S_j = \sum_{m=1}^{j} X_m \quad 1 \le j \le k$$

$$= (X_1, \ldots X_k)^{-1}(A) \in \mathcal{A}_k$$

Ex.

for suitable $A \subseteq \{-1, 1\}^k$

$\therefore$ σ_a is a stopping time.

• $L_a \equiv$ last time the walk hits 3 before time N.

$\{L_a = 4\} \notin \mathcal{A}_4$ ~ Ex construct a simple path

Theorem 1.8 :- [No Profit at any favourite moment]

let $\tau : \Omega \to \{0, 1, \ldots, N\}$ be a stopping time.

$$\mathbb{E}[S_\tau] = 0.$$