Recall :- • $\{Z_n\}_{n\geq 1}$ $E|Z_1| < \infty$ and

$E[Z_n | Z_{n-1}, \dots Z_1] = Z_{n-1}$ $\forall n \geq 2$.

• $T: \Omega \to \mathbb{N} \cup \{\infty\}$ is a stopping time if $k \in \mathbb{N}$, $\{T=k\} \in A_k$ (where A_k are event observable by time k)

$$1_{\{T=k\}} = \text{function of } Z_1, \dots Z_k$$

• [Theorem 3.4] $\{Z_n^T\}_{n\geq 1} = \{Z_{n\wedge T}\}_{n \geq 1}$ is also a martingale

[Corollary 3.5] $E[Z_n^T] = E[Z_1]$ $\forall n \geq 1$

[Optional Stopping Theorem]

$E[Z_T] = E[Z_1] \iff$ • $E[Z_n | T > n] \, \mathbb{P}(T > n) \longrightarrow 0$ as $n \to \infty$ —(OST)

• $E[|Z_T|] < \infty$.

Example (2b): • $X_i = \begin{cases} 2 & \text{wp } \frac{1}{2} \\ 0 & \text{wp } \frac{1}{2} \end{cases}$ $Z_n = \prod_{i=1}^{n} X_i$

(independent)

• $\mathbb{P}(Z_n = 2^n) = \frac{1}{2^n}$ & $\mathbb{P}(Z_n = 0) = 1 - \frac{1}{2^n}$

• $T = \min\{k \geq 1 \mid Z_k = 0\}$

- $Z_T = 0$, $E[Z_T] = 0$
- $Z_n^T \longrightarrow Z_T$ wp 1
- $1 = E[Z_1] = E[Z_n^T] \not\longrightarrow E[Z_T]$

Sufficient Conditions for (OST) in Theorem 3.7

(1) T is a bounded stopping time & $\{Z_n\}_{n\geq 1}$ is martingale $\Longrightarrow$ (OST)

(last class)

(2) $\mathbb{P}(T<\infty) = 1$, $\{Z_n\}_{n\geq 1}$ is a bounded martingale $\Longrightarrow$ (OST)

Proof: (i) $\exists M$: $E[Z_n \mid T>n] < M \quad \forall\, n\geq 1$

$\left(\mathbb{P}(T>n) \to 0 \text{ as } n\to\infty\right.$ $\Longrightarrow$ $E[Z_n \mid T>n]\,\mathbb{P}(T>n) \longrightarrow 0$ as $n\to\infty$

$\therefore\ \mathbb{P}(T<\infty) = 1$

(ii) $\{Z_n\}$ - bounded $\Longrightarrow$ $E|Z_T| < \infty$ $\quad\therefore$ (OST) ✓ $\square$

(3) $E[T]<\infty$ & $\exists\, c>0$ $|Z_n - Z_{n-1}| \leq c \quad \forall\, n\geq 2.$ $\Longrightarrow$ (OST)

Proof:

$\Longrightarrow \exists\, c>0$ such $\quad |Z_n| \leq |Z_n - Z_{n-1}| + |Z_{n-1}| \quad \forall\, n\geq 2$

$\leq c + |Z_{n-1}|$

(Inductively) $\Longrightarrow$ (growth) $\quad |Z_n| \leq (n-1)c + |Z_1| \quad \forall\, n\geq 2$ $\quad (*)$

From $(*)$: $\quad |E[Z_n \mid T>n]| \leq E\big[|Z_n| \,\big|\, T\geq n\big]$

$\leq (n-1)c + E\big[|Z_1| \,\big|\, T>n\big]$

$\therefore$ Condition (i) of (OST)

$|E[Z_n \mid T>n]\,P(T>n)| \leq c\,(n-1)\,\mathbb{P}(T>n) + E\big[|Z_1| \,\big|\, T>n\big]\,\mathbb{P}(T>n)$

(Ex.) $\longleftarrow = c\,(n-1)\,\mathbb{P}(T>n) + E[|Z_1|; T>n]$

(**) — $E|Z_1| < \infty$ & $\begin{cases} \mathbb{P}(T>n) \longrightarrow 0 \\ n\,\mathbb{P}(T>n) \longrightarrow 0 \end{cases} \Longleftarrow E[T] < \infty$

$\overset{\Longrightarrow}{(Ex)}$ $E[|Z_1|, T>n] \longrightarrow 0$ as $n \to \infty$

$(n-1)\,\mathbb{P}(T>n) \longrightarrow 0$ as $n \to \infty$

$\therefore$ (i) of (OST) holds.

(ii)

$$E|Z_T| = E \sum_{k=1}^{\infty} |Z_k| \cdot \mathbb{1}_{T=k}$$

$$\overset{(Ex)}{\underset{(*)}{\leq}} E \sum_{k=1}^{\infty} \left(c(k-1) + |Z_1|\right) \mathbb{1}(T=k)$$

$$\leq c \sum_{k=1}^{\infty} k\,\mathbb{P}(T=k) + E\Big(\sum_{k=1}^{\infty} |Z_1|\,\mathbb{1}_{T=k}\Big)$$

$$= c\,E[T] + E[|Z_1|] < \infty.$$

D

$\therefore$ (OST) holds

Corollary – OST: $T : \Omega \to \mathbb{N} \cup \{\infty\}$ is a stopping time wrt $\{Z_n\}_{n\geq 1}$

$\{Z_n\}$ is a martingale

If ①, ② or ③ hold $\Rightarrow E[Z_T] = E[Z_1]$

Example 4: (HW6 Q2 ⓑ)

$q = 1 - p \qquad 0 < p < 1$

X_i iid ; $i \geq 1$
$\mathbb{P}(X_i = 1) = p$
$\mathbb{P}(X_i = -1) = q$

$S_n = \sum_{i=1}^{n} X_i$

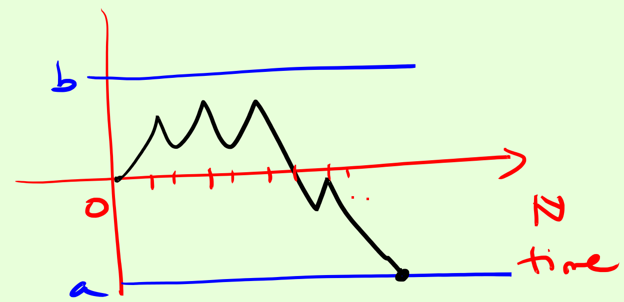
$S_0 = 0$

• $W_n = \left(\frac{q}{p}\right)^{S_n}$ is a martingale ;

$\sigma_a = \min\{ n \geq 1 \mid S_n = a \}$ $\qquad a < 0 < b$

$\sigma_b = \min\{ n \geq 1 \mid W_n = b \}$

Q:- $\quad \mathbb{P}(\sigma_a < \sigma_b) = ?$



Answer: Apply optional sampling Theorem to $\{W_n\}_{n \geq 1}$

• σ_a, σ_b are stopping times.

$T = \sigma_a \wedge \sigma_b$ — stopping time for S_n — [Shown in HW]

• [Ex] $\quad \mathbb{P}(T \geq N) = \mathbb{P}^{N}(T \geq N) \longrightarrow 0$ as $n \to \infty$ — (++)

(HW $p = q = \frac{1}{2}$)

• $E|W_T| \leq \left(\frac{q}{p}\right)^a + \left(\frac{q}{p}\right)^b := C_1$

• $\left| E(W_T \mid T > n)\, \mathbb{P}(T > n) \right| \leq E(|W_T| \mid T > n)\, \mathbb{P}(T > n)$

$\leq C_1\, \mathbb{P}(T > n)$ — (+)

Using (++) and (+) we have

$E(W_T \mid T > n)\, \mathbb{P}(T > n) \longrightarrow 0$ as $n \to \infty$.

$\therefore$ (OST) is satisfied by $\{W_n\}_{n \geq 1}$ & T

$\therefore$ Optional sampling Theorem

$$\Rightarrow \quad E[W_T] = E[W_1]$$

$$= \left(\frac{q}{p}\right)^{1} P(X_1 = 1) + \left(\frac{q}{p}\right)^{-1} P(X_1 = 0)$$

$$= \left(\frac{q}{p}\right) p + \left(\frac{p}{q}\right) q$$

$(\dagger)$ $\therefore$ $\boxed{E[W_T] = q + p = 1}$

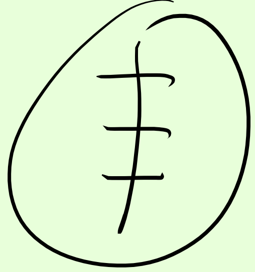
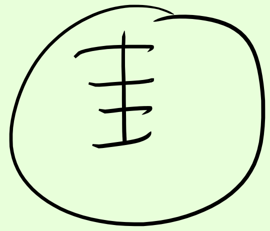
$(\ddagger)$ $\boxed{P(\sigma_a < \sigma_b) + P(\sigma_b < \sigma_a) = 1}$

$$E[W_T] = E[W_{\sigma_a} 1_{\sigma_a < \sigma_b}] + E[W_{\sigma_b} 1_{\sigma_b < \sigma_a}]$$

by $(\dagger)$ $\quad 1 = \left(\frac{q}{p}\right)^{a} P(\sigma_a < \sigma_b) + \left(\frac{q}{p}\right)^{b} P(\sigma_b < \sigma_a)$

$\left[W_n = \left(\frac{q}{p}\right)^{S_n}, \quad S_{\sigma_a} = a, \; S_{\sigma_b} = b\right]$

$(\ddagger\ddagger)$ $\boxed{1 = \left(\frac{q}{p}\right)^{a} P(\sigma_a < \sigma_b) + \left(\frac{q}{p}\right)^{b} P(\sigma_b < \sigma_a)}$

Ex:  and  we have

$$\mathbb{P}(\sigma_a < \sigma_b) = \frac{\left(\frac{q}{p}\right)^b - 1}{\left(\frac{q}{p}\right)^b - \left(\frac{q}{p}\right)^a}$$

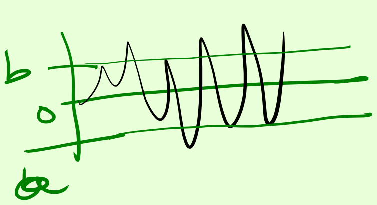
□

Kolmogorov Inequalities

[powerful strengthening] → Markov inequality → Chebyshev inequality → Weak law of large numbers.

- Kolmogorov Inequalities for martingales → [Strong law of large numbers] → Martingale convergence Theorem.

- In books / Courses :- typically these are understood by working on up-crossing & down crossing of a martingale.

" 

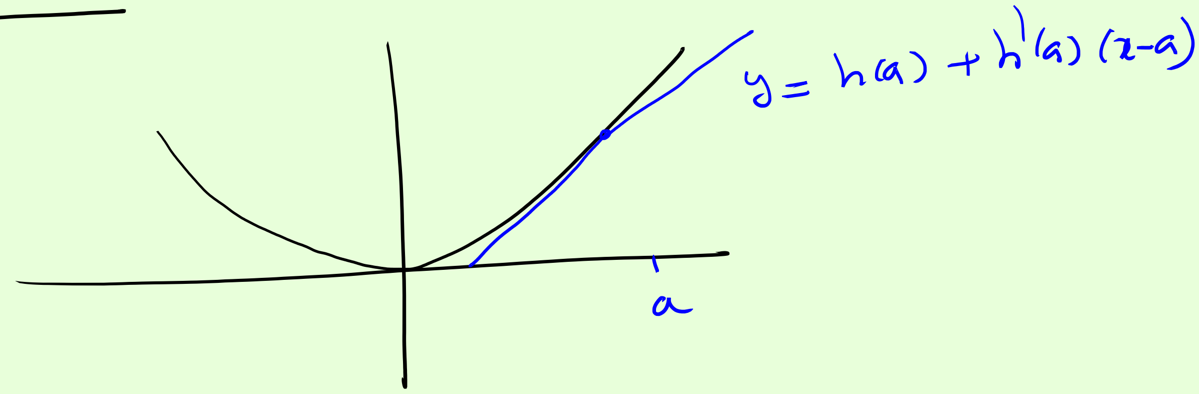
"

- Enlarge our view of process other than martingales

Convex functions – Digression

$h: \mathbb{R} \to \mathbb{R}$

- $h(x) = x^2$



Observe :- $h(x) \geq h(a) + h'(a)(x-a)$ $\forall x \in \mathbb{R}$ $\forall a \in \mathbb{R}$

Definition 3.6 :- $h: \mathbb{R} \to \mathbb{R}$ is said to be convex if $\forall a \in \mathbb{R}$ $\exists c \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$

$$h(x) \geq h(a) + c(x-a)$$

Ex: $h: \mathbb{R} \to \mathbb{R}$ $h''(\cdot) \geq 0 \Rightarrow h$ is convex.

Jensen's Inequality :- X is a random variable $E|X| < \infty$. $h: \mathbb{R} \to \mathbb{R}$ be a convex function

$$E(h(X)) \geq h(E(X))$$

Proof: $a = E[X]$, h is convex

$$\Rightarrow \exists c \in \mathbb{R}: h(X) \geq h(a) + c(X-a)$$

$$\Rightarrow E[h(X)] \geq E(h(a)) + c(E[X]-a)$$

$$\Rightarrow E[h(X)] \geq h(a)$$

$$\Rightarrow E(h(X)) \geq h(E(X)) \quad \square$$

Corollary to Jensen's: A - event and X r.v. $E|X| < \infty$

$h: \mathbb{R} \to \mathbb{R}$ is convex

$$h(E(X|A)) \leq E(h(X)|A) \quad \text{— JI}$$

Proof: Ex - imitate previous proof. □

Example 6: • let $\{Z_n\}_{n\geq 1}$ be a martingale.

• $h: \mathbb{R} \to \mathbb{R}$ convex function $E|h(Z_n)| < \infty$ $\forall n \geq 1$

$$\bullet\ E[h(Z_n) | Z_{n-1}, \ldots Z_1] \overset{(JI)}{\geq} h(E(Z_n | Z_{n-1}, \ldots Z_1))$$

$$\overset{\text{martingale}}{=} h(Z_{n-1})$$

Then $\{h(Z_n)\}_{n\geq 1}$ — sub-martingale.

Definition 3.7: $\{Z_n\}_{n\geq 1}$ sequence of random variables is said to be a sub-martingale if

$E|Z_n| < \infty$ and $E[Z_n | Z_{n-1}, \ldots, Z_1] \geq Z_{n-1}$

Example 6 (Contd): • $h(x) = x^2$ $h: \mathbb{R} \to \mathbb{R}$

$\{Z_n\}_{n\geq 1}$ is a martingale & $E[Z_n^2] < \infty \Rightarrow \{Z_n^2\}_{n\geq 1}$ is a sub-martingale.

- $\{S_n\}_{n\geq 1}$ — Simple random walk. — $h(x) = e^{rx}$

$E|h(S_n)| < \infty \implies \{e^{rS_n}\}_{n\geq 1}$ is a sub-martingale

Recall: $\{e^{rS_n - ng(r)}\}_{n\geq 1}$ — was shown to be a martingale $\quad g(r) = E e^{rX_1}$

Theorem 3.8 :- [Kolmogorov's sub martingale Inequality] $\{Z_n\}_{n\geq 1}$ be a non-negative sub-martingale. Then for any positive integer m and any $a>0$

$$\mathbb{P}\left(\max_{1\leq i\leq m} Z_i \geq a\right) \leq \frac{E[Z_m]}{a}$$

(Compare with markov Inequality : $\mathbb{P}(Z_i \geq a) \leq \frac{E[Z_i]}{a}$)

Corollary 3.9 :- let $\{Z_n,\ n\geq 1\}$ be a martingale with $E[Z_n^2] < \infty$ $\forall n\geq 1$. Then for any positive integer m and $b>0$

$$\mathbb{P}\left(\max_{1\leq n\leq m} |Z_n| \geq b\right) \leq \frac{E[Z_m^2]}{b^2}$$

Proof: $\{Z_n\}_{n\geq 1}$ is a martingale

(Example 6) $\implies \{Z_n^2\}_{n\geq 1}$ is a sub-martingale

and by definition it is non-negative

Apply Theorem 3.7:

$a = b^2$

$$\mathbb{P}\left(\max_{1\le i\le m} Z_i^2 > b^2\right) \le \frac{\mathbb{E} Z_m^2}{b^2}$$

$$\Rightarrow \quad \mathbb{P}\left(\max_{1\le i\le m} |Z_i| > b\right) \le \frac{\mathbb{E} Z_m^2}{b^2} \quad \circ$$