Discussion on Conditional Expectation :-

S-countable

$\Omega = S^{\mathbb{N}}$; $X_n(\omega) = \omega_n$ $n \in \mathbb{N} \cup \{\infty\}$

$\{X_n\}_{n\geq 1}$ — $\mathcal{A}_n \equiv$ events observable by time n

— $\{\omega \in \Omega \mid \omega_1 = o_1, \dots \omega_n = o_n\}$

$o_i \in S$.

$$E[X_n \mid X_1 = x_1 \dots X_{n-1} = x_{n-1}] \overset{\checkmark}{=} \sum_{k \in \text{Range}(X_n)} k\,\mathbb{P}(X_n = k \mid X_1 = x_1 \dots X_{n-1} = x_{n-1})$$

$$\Downarrow \Uparrow$$

$E[X_n \mid X_1 \dots X_{n-1}] \equiv$ random variable

$\cdot\; f(x_1, \dots x_{n-1}) = E[X_n \mid X_1 = x_1 \dots X_{n-1} = x_{n-1}]$ } random variable

$Y_n =$ "$f(X_1, \dots, X_{n-1})$" $=: E[X_n \mid X_1 \dots X_{n-1}]$

Prescription of Y_n : (Discrete random variables)

$E|X_n| < \infty$

$$\equiv \quad Y_n := \sum_{x_i \in \text{Range}(X_i)} E[X_n \mid X_1 = x_1 \dots X_{n-1} = x_{n-1}]\, \underbrace{\mathbb{1}(x_i = X_i(\omega), 1 \leq i \leq n-1)}$$

$\mathbb{1}_{\dots} \begin{matrix} = 0 \\ = 1 \end{matrix} \equiv$ in $\mathcal{A}_{n-1}$

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$E[X_n \mid \mathcal{A}_{n-1}]$

Y_n is "predictable" w.r.t. $\mathcal{A}_{n-1}$

(i) $\{Y_n = c\} \in \mathcal{A}_{n-1}$

Def: X - random variable, A is an event

$$E[X\, \mathbb{1}_A] = \sum_{k \in \text{Range}(X)} \mathbb{P}(X = k, A)\, k \quad \checkmark$$

(ii) $A \in \mathcal{A}_{n-1}$: $E[Y_n \mathbb{1}_A] = E[X_n \mathbb{1}_A]$ (Ex)

Recall:- $\{Z_n\}_{n\ge1}$ $\quad E|Z_n|<\infty \quad \forall n\ge 1$ — Assume

Z_n Discrete r.v.

• $$E[Z_n | Z_{n-1},\dots,Z_1] = \sum_{\substack{z_i\in \mathrm{Range}(Z_i)\\ 1\le i\le n-1}} E[Z_n | Z_1=z_1,\dots, Z_{n-1}=z_{n-1}]\, \mathbb{1}_{B^{z}_{n-1}}$$

where $B^{z}_{n-1} = \{Z_{n-1}=z_{n-1}, \dots, Z_1=z_1\}$; $\displaystyle\bigcup_{z\in\prod_{i=1}^{n-1}\mathrm{Range}\, Z_i} B^{z}_{n-1} = \Omega$.

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$E[Z_n | A_{n-1}]$ where $A_{n-1} \equiv$ events observable by time $n-1$ w.r.t $\{Z_k\}_{k\ge1}$

• [Tower Property]

If $\{\phi,\Omega\} = B_0 \subseteq B_1 \dots \subseteq B_n$

& $B_k \subseteq A_k \quad \forall k\ge1$

-(TP) $$E[\,E[Z_n|A_k]\,|\,B_k] = E[Z_n|B_k] \quad \forall n\ge1,\ k<n$$

• [observable Property]

(O) $$E[Z_n | A_n] = Z_n$$

Definition of Martingale :- - $\{X_n\}_{n\ge1}$ is a m'ng

- $E|X_n|<\infty \quad \forall n\ge1$
- $E[X_n|X_{n-1},\dots X_1] = X_{n-1}$ $\longleftrightarrow$ • $\boxed{E|X_n|<\infty \quad E[X_n|A_{n-1}] = X_{n-1}}$

$\{\xi_n\}_{n\ge1}$ and B_n - events observable by time n by $\{\xi_i\}_{i\ge1}$ are st

$|||$

"$f(X_n)$ for some f"

$$B_n \subseteq A_n \quad \forall n\ge1$$

• It is enough to show $E|\xi_n|<\infty$ and $E[\xi_n|A_{n-1}] = \xi_{n-1}$ $\forall n\ge2$ for determining whether $\{\xi_n\}_{n\ge1}$ is a martingale or not.

Definition 3.3 :- • A stopping time $T : \Omega \to \{1, 2, \dots\} \cup \{\infty\}$ is called a stopping wrt $\{Z_n\}_{n\geq 1}$ if

$\{T = n\}$ is an observable event by time n.

$(\Omega, \mathcal{F}, \mathbb{P})$ — Probability space exist where $\{Z_n\}_{n\geq1}$ is defined

i.e $\{T=n\} \in A_n$

• or $\mathbb{1}_{\{T=n\}} \equiv$ function of $Z_1, \dots, Z_n$

• stopped process

$$Z_n^T = \begin{cases} Z_n & n \leq T \\ Z_T & n > T \end{cases} \equiv Z_{n \wedge T}$$

where $Z_{n\wedge T}(\omega) := Z_{n \wedge T(\omega)}(\omega)$.

Theorem 3.4: Given a sequence of (discrete) random variables $\{Z_n\}_{n\geq 1}$ and a stopping time $T : \Omega \to \{1, \dots\} \cup \{\infty\}$, then the stopped process is $\{Z_n^T\}_{n\geq1}$ is a martingale if $\{Z_n\}_{n\geq 1}$ is a martingale.

Proof :- Method A :- Q3 of HW 6 — [Uses the fact "Game System" is a martingale] — Ex.

To show (i) $E|Z^{n}_{T}| < \infty$

Proof of (i) :- Using definition of $\{Z^{n}_{T}\}_{n\geq 1}$ we have

$\forall n \geq 1$; $E|Z^T_n| = \sum_{i=1}^{n-1} E[|Z^T_n| \, 1_{T=i}] + E[|Z^T_n| \, 1_{T\geq n}]$

Assume W.L.O.G :- $P(T=i) > 0 \ \forall i = 1 .. n-1$ & $P(T \geq n) > 0$ [Ex]

$\left(Z^T_n = Z_{\underbrace{n\wedge T}_{\min(n,T)}} \Rightarrow \right) = \sum_{i=1}^{n-1} E[|Z_i| \, 1_{T=i}] + E[|Z_n| \, 1_{T\geq n}]$

$\left(1_{(T=i)} \leq 1 \Rightarrow \right) \leq \sum_{i=1}^{n-1} E|Z_i| + E|Z_n|$

$< \infty$ as $\{Z_n\}_{n\geq 1}$ is a martingale

Show (ii) $E[Z^T_n \mid Z^T_{n-1} .. Z^T_1] = Z^T_{n-1}$

Suppose $\mathcal{A}_n$ are the observable events by time n w.r.t $\{Z_k\}_{k\geq 1}$

Suppose $\mathcal{B}_n$ are the observable events by time n w.r.t $\{Z^T_k\}_{k\geq 1}$.

$\mathcal{B}_n \subseteq \mathcal{A}_n$.

[HW 5, Q2 - Same proof for this set up as well.] $\because$ T is a stopping time $\Rightarrow$ $\{T=k\} \in \mathcal{A}_k$

$Z^T_n = Z_{n\wedge T} \equiv f(\underbrace{Z_n, T}_{\text{predictable w.r.t } \mathcal{A}_n})$

By (TP) it is enough to show

$E[Z^T_n \mid \mathcal{A}_{n-1}] = Z^T_{n-1}$

$$Z_n^T = \sum_{k=1}^{n-1} Z_n^T \mathbb{1}_{T=k} + Z_n^T \mathbb{1}_{T \geq n}$$

$$= \sum_{k=1}^{n-1} Z_k \mathbb{1}_{T=k} + Z_n \mathbb{1}_{T \geq n}$$

$$E\left[Z_n^T \mid A_{n-1}\right] = E\left[\sum_{k=1}^{n-1} Z_k \mathbb{1}_{T=k} + Z_n \mathbb{1}_{T \geq n} \,\middle|\, A_{n-1}\right]$$

Conditional Expectation is linear $\longleftarrow$

$$= \sum_{k=1}^{n-1} \underbrace{E\left[Z_k \mathbb{1}_{T=k} \mid A_{n-1}\right]} + E\left[Z_n \mathbb{1}_{T \geq n} \mid A_{n-1}\right]$$

O $\longleftarrow$

$$= \sum_{k=1}^{n-1} Z_k \mathbb{1}_{T=k} + E\left[Z_n \mathbb{1}_{T \geq n} \mid A_{n-1}\right]$$

$A_k \subseteq A_{n-1}$

$k \leq n-1$

$\left(\begin{array}{l} \{T=k\} \in A_k \\ \{Z_k = c\} \in A_k \end{array} \right)$

$\left(\begin{array}{l} \bullet\ \{T \leq n-1\} \in A_{n-1} \\ \bullet\ \{T \geq n\}^c = \{T \leq n-1\} \in A_{n-1} \end{array} \right)$

$\Rightarrow T \geq n \in A_{n-1}$

$$= \sum_{k=1}^{n-1} Z_k \mathbb{1}_{T=k} + \mathbb{1}_{T \geq n} E\left[Z_n \mid A_{n-1}\right]$$

$$= \sum_{k=1}^{n-1} Z_k \mathbb{1}_{T=k} + \mathbb{1}_{T \geq n} Z_{n-1}$$

$$= \sum_{k=1}^{n-2} Z_k \mathbb{1}_{T=k} + \mathbb{1}_{T \geq n-1} Z_{n-1}$$

$$= Z_{n-1}^T$$

□

Corollary 3.5 :- $T : \Omega \to \mathbb{N} \cup \{\infty\}$ is a stopping time w.r.t. $\{Z_n\}_{n\geq 1}$ and Z_n is a martingale

then $$E[Z_n^T] = E[Z_1]$$ $\square$

Proof : $\{Z_n^T\}_{n\geq 1}$ is a martingale by Theorem 3.4 & use Corollary 3.2. $\square$

Optional Stopping (time) Theorem :-

Recall :- Example 2(b) :- $X_i = \begin{cases} 2 & \text{wp } \frac{1}{2} \\ 0 & \text{wp } \frac{1}{2} \end{cases}$ $\quad Z_n = \prod_{i=1}^{n} X_i$

$i \geq 1$, (independent)

- $\mathbb{P}(Z_n = 2^n) = \frac{1}{2^n}$ $\quad \mathbb{P}(Z_n = 0) = 1 - \frac{1}{2^n}$ $\quad \forall n \geq 1$

- $\varepsilon > 0$ be given $\quad 0 \leq \mathbb{P}(|Z_n| > \varepsilon) = \mathbb{P}(Z_n = 2^n) = \frac{1}{2^n}$ $\quad \forall n \geq 1$

$\therefore \mathbb{P}(|Z_n| > \varepsilon) \to 0$ as $n \to \infty$

$\Rightarrow Z_n \xrightarrow{P} 0$ as $n \to \infty$.

Stopped process. $T = \min\{k \geq 1 \mid Z_k = 0\}$

$Z_T^n = Z_n \quad \forall n \geq 1$, $\underbrace{Z_T = 0 \text{ wp } 1}_{\text{by definition of } T}$

(If the process has not stopped $Z_n^T = Z_n$ otherwise 0)

$\Downarrow$

Theorem 3.3: $E[Z_n^T] = E[Z_1] = 1$, $E[Z_T] = 0$

$\therefore$

$Z_n^T \to Z_T$ as $n \to \infty$ w.p.1 but $\lim_{n\to\infty} E[Z_n^T] \neq E[Z_T]$

Theorem 3.6 let T be a stopping time

$T: \Omega \to \mathbb{N}$ $\leftarrow$ [Assumption]

wrt a martingale $\{Z_n\}_{n\geq 1}$

$$E[Z_T] = E[Z_1] \iff E(Z_n \mid T \geq n)\, \mathbb{P}(T \geq n) \to 0 \text{ as } n \to \infty$$

Proof:- • We know $\{Z_n^T\}_{n\geq 1}$ is a martingale

$\therefore$ $E[|Z_n^T|] < \infty \quad \forall n \geq 1$ — ⊛

• $Z_n^T = Z_{T\wedge n} \longrightarrow Z_n$ as $n\to\infty$ w.p. 1 — (Ex)

($\because$ $\mathbb{P}(T<\infty) = 1$)

Using ⊛

$$E[Z_n^T] = \sum_{i=1}^{n-1} E[Z_n^T ; T=i] + E[Z_n^T ; T \geq n]$$

(wlog $\mathbb{P}(T=i) > 0$ $\forall i \in \mathbb{N}$) $\leftarrow$

$$= \sum_{i=1}^{n-1} E[Z_n^T \mid T=i]\, \mathbb{P}(T=i) + E(Z_n^T \mid T \geq n)\, \mathbb{P}(T \geq n)$$

$$= \sum_{i=1}^{n-1} E[Z_T \mid T=i]\, \mathbb{P}(T=i) + E(Z_n \mid T \geq n)\, \mathbb{P}(T\geq n) \quad \text{— ⊛⊛}$$

$$E[Z_T] = \sum_{i=1}^{\infty} E[Z_T ; T=i] = \sum_{i=1}^{\infty} E[Z_T \mid T=i]\, \mathbb{P}(T=i)$$

$\therefore$ If $E(Z_n \mid T \geq n)\, \mathbb{P}(T \geq n) \longrightarrow 0$ as $n \to \infty$

then using ① $E[Z_n^T] = E[Z_1]$

& ② $\lim_{n\to\infty} \sum_{i=1}^{n-1} E(Z_T \mid T=i)\, \mathbb{P}(T=i) = E[Z_T]$

③ Taking limits in (xx)

$$\Rightarrow \quad E[Z_1] = E[Z_T] \qquad \square$$

Examples where hypothesis of Theorem 3.6 are satisfied

- $E(Z_n \mid T \geq n)\, \mathbb{P}(T \geq n) \longrightarrow 0$ as $n \to \infty$

- $\{Z_n\}_{n \geq 1}$ is a martingale & T is a bounded stopping

$\exists N : \quad \mathbb{P}(T > N) = 0$

$\therefore$

$\Rightarrow \quad E[Z_n ; T \geq n] = 0 \quad \forall\ n \geq N+1$

$\Rightarrow$ " $\underbrace{E(Z_n \mid T \geq n)}_{\text{only valued}}\, \mathbb{P}(T \geq n)$ " $= 0 \qquad n \geq N+1$

if $\mathbb{P}(T \geq n) > 0$ $\qquad \square$