Recall :-

Markov chain notion of Conditioning past

[lemma 3.2] $\{Z_n\}_{n\geq 1}$ is a martingale

$$E[Z_n \mid Z_i, Z_{i-1}, \dots Z_1] = Z_i \qquad \forall\, n \geq i \geq 1$$

[Corollary] $i=1$ in lemma to obtain

$$E[Z_n] = E[Z_1] \qquad \forall\, n \geq 1$$

Definition 3.3 :- $(\Omega, \mathcal{A}, \mathbb{P})$ be a Probability space. Let $\{Z_n\}_{n\geq 1}$ be a sequence of random variables on it. A $T: \Omega \to \mathbb{N} \cup \{\infty\}$ is called a stopping time for $\{Z_n\}_{n\geq 1}$ if

$\{T = k\}$ is an observable event by time k. $\equiv A_k$

- [Remark] $\mathbb{1}(T=k)$ is a function of $Z_1, Z_2, \dots Z_k$ $\forall\, k \in \text{Range}(T)$

[Recall] - Compare and see above matches definition of stopping time given for S.R.W.

Definition 3.4 :- For any stopping time T we define the stopped process

$$Z_n^T := \begin{cases} Z_n & n \leq T \\ Z_T & n > T \end{cases} \equiv Z_{n \wedge T}$$

Notation: $Z_T(\omega) = Z_{T(\omega)}(\omega)$

- [Seen Examples before] → Gambler's Ruin M.C. - stopped the chain when capital of any one player hits 0.

- Created sort of absorption sites for M.C.

- $\sigma_a = \min\{n \geq 1 \mid S_n = a\}$ $\qquad S_n^{\sigma_a} = \begin{cases} S_n & n \leq \sigma_a \\ a & n > \sigma_a \end{cases}$

Theorem 3.5 :- Given a sequence $\{Z_n\}_{n\geq 1}$ and a stopping time $T: \Omega \to \mathbb{N}$ w.r.t. the stopped process $\{Z_n^T\}_{n\geq 1}$ is a martingale if $\{Z_n\}_{n\geq 1}$ is a martingale.

[Intuitive Proof] $E[Z_n^T \mid Z_{n-1}^T, \dots Z_1^T] = Z_{n-1}^T$ [To show]

Take $Z_n = z_n$, $Z_{n-1} = z_{n-1}$, $\dots$, $Z_1 = z_1$

Given a sample path :- $(z_{n-1}, \dots, z_1) \longrightarrow$ will determine the behaviour of $T \rightarrow$ - $T \geq n$, $z_n^T = z_n$

- $T < n$, $z_n^T = z_{n-1}$

$\Rightarrow$ $\therefore E[Z_n^T \mid Z_{n-1}^T = z_{n-1}^T, \dots Z_1^T = z_1^T] = z_{n-1}$ $\square$

Conditional Expectation [Revisit |← "Defining it precisely" - Discrete r.v.]

$\{X_n\}_{n\geq 1}$ - sequence of discrete random variables on $(\Omega, \mathcal{A}, \mathbb{P})$.

Range(X_n) - $\subseteq \mathbb{R}$ $\forall n \geq 1$

- Time as $n\geq 1$:- $\mathcal{A}_0 = \{\phi, \Omega\}$

- $A \subseteq \Omega$ is an event observable by time n if it can be written as a union of basic events of

$\{\omega \in \Omega \mid X_1(\omega) = x_1, X_2(\omega) = x_2, X_n(\omega) = x_n\}$, $x_i \in$ Range(X_i) $1 \leq i \leq n$.

$\mathcal{A}_n \equiv$ set of all observable events by time n.

Y, Z are two discrete random variables on $(\Omega, \mathcal{A}, \mathbb{P})$

- $E[Z \mid X_1 = x_1] = \sum_{k \in \text{Range}(Z)} k \, \mathbb{P}(Z = k \mid X_1 = x_1)$ — $\mathbb{P}(X_1 = x_1) > 0$

- $E[Z \mid X_n = x_n, \dots, X_1 = x_1] = \sum_{k \in \text{Range}(Z)} k \, \mathbb{P}(Z = k \mid X_n = x_n, \dots X_1 = x_1)$ — $\mathbb{P}(X_n = x_n \dots X_1 = x_1) > 0$

- understanding of $E[Z \mid X_1, \dots, X_n]$ $\quad f : \prod_{i=1}^{n} \text{Range}(X_i) \to \mathbb{R}$

$f(x_1, \dots, x_n) = E[Z \mid X_1 = x_1, \dots, X_n = x_n]$

- $E[Z \mid X_1 \dots X_n] \equiv$ " $f(X_1, \dots X_n)$ ".

Definition 3.6:- let $\{X_n\}_{n \geq 1}$ be a sequence of discrete random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. let Z be any discrete r.v.

The conditional expectation of Z given $X_1 \dots X_n$ is a r.v. $E[Z \mid X_1 \dots X_n] : \Omega \to \mathbb{R}$

given by

$$E[Z \mid X_1, \dots, X_n] := \sum_{x \in \text{Range}(X)} E[Z \mid X_1 = x_1 \dots X_n = x_n] \, \mathbb{1}_{(X_1 = x_1, \dots, X_n = x_n)}$$

$x = (x_1, \dots x_n)$

$\text{Range}(X) = \text{Range}(X_1, \dots, X_n)$

CE — $E[Z \mid \mathcal{A}_n]$

$\mathcal{A}_n =$ observable events upto time n by $\{X_k\}_{k \geq 1}$

— filtration — $\mathcal{A}_n \subseteq \mathcal{A}_{n-1}$

Calculation :

X - Discrete random variable. [Motivation question is wrt Home work -4]

$Y = X^2$ [$Y=f(X)$ for some f] Calculation will apply

$\{X=c\}$ $c \in Range(X)$

$\mathcal{A}_X \equiv$ events observable by X ⇝ union of basic event ~

$\{Y=d\}$ $d \in Range(Y)$

$\mathcal{A}_Y =$ events observable by Y.

Ex — ∘ $\mathcal{A}_Y \subseteq \mathcal{A}_X$

∘ $E[X \mid X=c] = c$

OE — • $E[X \mid \mathcal{A}_X] := \sum_{c \in Range(X)} E[X \mid X=c] \, \mathbb{1}_{X=c}$

$$= \sum_{c \in Range(X)} c \, \mathbb{1}_{X=c} = X$$

$$E[Z \mid \mathcal{A}_X] := \sum_{c \in Range(X)} E[Z \mid X=c] \, \mathbb{1}(X=c)$$

Tower Property - [Abstract Notion - Based on observable events]

$\boxed{E|Z| < \infty}$

$$E\left[E[Z \mid \mathcal{A}_X] \mid \mathcal{A}_Y \right] =$$

TP

$$E\left[\sum_{c \in Range(X)} E[Z \mid X=c] \, \mathbb{1}(X=c) \;\middle|\; \mathcal{A}_Y \right]$$

$$= E\left[\sum_{c \in Range(X)} \sum_{k \in Range(X)} k \, \mathbb{P}(Z=k \mid X=c) \, \mathbb{1}(X=c) \,\middle|\, \mathcal{A}_Y \right]$$

absolute convergent series

$$= \sum_{c \in Range(X)} \sum_{k \in Range(X)} k \, \mathbb{P}(Z=k \mid X=c) \, E[\mathbb{1}(X=c) \mid \mathcal{A}_Y]$$

$$= \sum_{c \in Range(X)} \sum_{k \in Range(X)} k \, \mathbb{P}(Z=k \mid X=c) \sum_{m \in Range(Y)} E[\mathbb{1}(X=c) \mid Y=m] \, \mathbb{1}(Y=m)$$

$$= \sum_{c \in Range(X)} \sum_{k \in Range(X)} k \, \mathbb{P}(Z=k \mid X=c) \sum_{m \in Range(Y)} \mathbb{P}(X=c \mid Y=m) \, \mathbb{1}(Y=m)$$

$$Y = X^2$$

$$= \sum_{c \in Range(X)} \sum_{k \in Range(X)} k \, \mathbb{P}(Z=k \mid X=c) \sum_{m \in Range(Y)} \frac{\mathbb{P}(X=c, Y=m)}{\mathbb{P}(Y=m)} \, \mathbb{1}(Y=m)$$

$$= \sum_{c \in Range(X)} \sum_{k \in Range(X)} k \sum_{m \in Range(Y)} \frac{\mathbb{P}(Z=k, X=c)}{\mathbb{P}(X=c)} \cdot \frac{\mathbb{P}(X=c, Y=m)}{\mathbb{P}(Y=m)} \, \mathbb{1}(Y=m)$$

$c = +\sqrt{m}$ or $c = -\sqrt{m}$

(Ex.)

$$= \sum_{k \in Range(X)} k \sum_{m \in Range(Y)} \frac{\mathbb{P}(Z=k \;\; Y=m)}{\mathbb{P}(Y=m)} \, \mathbb{1}(Y=m)$$

$$= \sum_{m \in Range(Y)} E[Z \mid Y=m] \, \mathbb{1}_{Y=m}$$

$$:= E[Z \mid \mathcal{A}_Y]$$

Observation : Motivation HW5 : 2

- $\{X_n\}_{n\geq 1}$ random discrete variables .... $\mathcal{A}_n$ - observable events by time n

$\mathcal{A}_n \subseteq \mathcal{A}_{n+1} \subseteq \cdots$

filtration 1 $\{\Omega, \phi\} = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots$

- $\{\xi_n\}_{n\geq 1}$ - discrete random variables .... $\mathcal{B}_n$ - observable event

$\mathcal{B}_n \subseteq \mathcal{B}_{n+1} \subseteq \cdots$

filtration 2 : $\{\Omega, \phi\} = \mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \cdots$

Suppose : $\mathcal{B}_n \subseteq \mathcal{A}_n$ — $\forall n \geq 1$

Z - discrete random variable

TP $\overset{Ex}{\Longrightarrow}$ $E[E[Z \mid \mathcal{A}_k] \mid \mathcal{B}_k] = E[Z \mid \mathcal{B}_k]$

Apply to HW5 Q2 :

$(S_n)_{n\geq 1}$ — simple random walk $\left(S_n = \sum_{i=1}^{n} X_i\right)$ $S_0 = 0$ $(S_n - S_{n-1} = X_n)$ ..... $\mathcal{A}_n$ -- observable by $\{X_n\}_{n\geq 1}$

$\xi_n = S_n^2 - n$ .... $\{\Omega, \phi\} = \mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \cdots \mathcal{B}_n \subseteq \cdots$

Ex : $\mathcal{B}_n \subseteq \mathcal{A}_n$ $\forall n \geq 1$

To show ξ_n is martingale : $E|\xi_n| < \infty$ - to be checked

$$E[\xi_n \mid \xi_{n-1}, \dots, \xi_1] = E[\xi_n \mid \mathcal{B}_n]$$

$\mathcal{B}_n \subseteq \mathcal{A}_n$

$$\overset{(TP)}{=} E[E[\xi_n \mid \mathcal{A}_n] \mid \mathcal{B}_n]$$

$$= E[E[\xi_n \mid S_{n-1}, \dots S_1] \mid \mathcal{B}_n]$$

$$\overset{HW5}{=} E[\xi_{n-1} \mid \mathcal{B}_n]$$

(ξ_{n-1} is observable entirely by events in $\mathcal{B}_n$) $= \xi_{n-1}$

Definition of Martingale :- - $\{X_n\}_{n \geq 1}$ is a martingale

- $E|X_n| < \infty \quad \forall n \geq 1$
- $E[X_n \mid X_{n-1}, \dots X_1] = X_{n-1}$

$$\boxed{\begin{gathered} E|X_n| < \infty \\ E[X_n \mid \mathcal{A}_{n-1}] = X_{n-1} \end{gathered}}$$

$\{\xi_n\}_{n \geq 1}$ and $\mathcal{B}_n$ - events observable by time n by $\{\xi_i\}_{i \geq 1}$ are s.t.

$\equiv$ "$f(X_n)$ for some f"

$$\mathcal{B}_n \subseteq \mathcal{A}_n \quad \forall n \geq 1$$

To show: ξ_n is a martingale

$E|\xi_n| < \infty$

$E[\xi_n \mid \xi_{n-1}, \dots \xi_1] = \xi_{n-1}$

$\Leftrightarrow$

$E|\xi_n| < \infty$

$E[\xi_n \mid \mathcal{B}_{n-1}] = \xi_{n-1}$

TP

$\Leftrightarrow$

$E|\xi_n| < \infty$ ✓

$E[\xi_n \mid \mathcal{A}_{n-1}] = \xi_{n-1}$