Recall:

Definition 3.1 A sequence of random variables $\{Z_n : n \geq 1\}$ is said to be a Martingale if

$E|Z_n| < \infty, \; \forall n \geq 1$

$$E[Z_n \mid Z_{n-1}, \dots, Z_1] = Z_{n-1} \quad \forall n \geq 2. \qquad \text{— (1)}$$

Correction addition

- Understand — (1)

View A

$$E[Z_n \mid Z_{n-1} = z_{n-1}, \dots, Z_1 = z_1] = z_{n-1}$$

$\underbrace{z_{n-1}, \dots z_1 \in \mathbb{R}}_{\text{Range of } Z_{n-1}, \dots Z_1}$

✓ If all Z_i's were discrete r.v.

✓ If all Z_i's were continuous r.v with joint pdf.

View B

$$f(z_{n-1}, \dots, z_1) = E[Z_n \mid Z_{n-1} = z_{n-1} \dots Z_1 = z_1]$$

(1) as r.v

$$Y_n = f(Z_{n-1}, Z_{n-2}, \dots, Z_1)$$

Y_n has Properties:

Requires mathematics outside the scope of the course

(i), (ii)

(i) $A = \{Z_{n-1} = z_{n-1}, \dots, Z_1 = z_1\}$

$$Y_n(\omega) = E[Z_n \mid Z_{n-1} = z_{n-1}, \dots Z_1 = z_1] \quad \forall \omega \in A$$

(ii) $A = \{Y_n \in B\}$ or $\{Y_n = c\}$

A or $\{Y_n = c\}$ are in "functions of $Z_{n-1}, \dots Z_1$"

$\in \mathcal{A}_{n-1} \equiv$ "observable by time $n-1$"

$$\text{"} E[Z_n \mid \mathcal{A}_{n-1}] = Y_n \text{"}$$

Example 1 :-

$S_n = \sum_{i=1}^{n} X_i$, X_i i.i.d X

replace can: $EX=0$, $E|X|<\infty$

$$X = \begin{cases} 1 & \text{wp } \frac{1}{2} \\ -1 & \text{wp } \frac{1}{2} \end{cases}$$

$\forall n \geq 1 \quad E|S_n| = E\left|\sum_{i=1}^{n} X_i\right| \leq n E|X| < \infty$

Triangle inequality & linearity of Expectation

$$E[S_n \mid S_{n-1}, \dots, S_1] = E[X_n + S_{n-1} \mid S_{n-1}, \dots, S_1]$$

(*) linearity of Conditional Expectation

$$= E[X_n \mid S_{n-1}, \dots, S_1] + E[S_{n-1} \mid S_{n-1}, \dots, S_1]$$

X_n is independent of $S_{n-1}, \dots S_1$

$$= E[X_n] + S_{n-1}$$

Reason :- X, Y are independent & discrete r.v's

$$E[X \mid Y=y] = \sum_{k \in Range(X)} k \, \mathbb{P}(X=k \mid Y=y)$$

independence

$$= \sum_{k \in Range(X)} k \, \mathbb{P}(X=k)$$

$$= E[X]$$

$$E[X \mid Y] = E[X]$$

Reason: X, Y, Z are discrete r.v's

$X = f(Y, Z)$

$E[X \mid Y, Z] = ?$

$$E[X \mid Y=y, Z=z]$$

$$= \sum_{k \in Range(X)} k \, \mathbb{P}(X=k \mid Y=y, Z=z)$$

$$= \sum_{k \in Range(X)} k \, \mathbb{P}(f(Y,Z)=k \mid Y=y, Z=z)$$

Suppose $u = f(y, z)$

$$= u \cdot \frac{\mathbb{P}(f(y,z)=u, Y=y, Z=z)}{\mathbb{P}(Y=y, Z=z)}$$

$$= u = f(y, z) = \cdot$$

$$E[X \mid Y, Z] = X$$

$$E[S_n \mid S_{n-1}, \dots, S_1] = E[X] + S_{n-1} = S_{n-1}$$

$\therefore \{S_n\}_{n\geq 1}$ is a martingale.

Key Results:- $\rightarrow$ T- stopping time $\quad E[S_T] = 0$, $E[S_T^2] = E[T]$

$\rightarrow$ $\tau = \min\{k \geq 1 \; S_k \notin [a,b]\}$ $\Rightarrow E\tau = -ab$

$a < 0 < b$, $S_0 = 0$

Example 2:- X_i iid $EX_i = 1$ $\quad E|X| < \infty$ $\leftarrow$ (addition)

$$Z_n = \prod_{i=1}^{n} X_i$$

$\forall n \geq 1 \quad E|Z_n| = E\left|\prod_{i=1}^{n} X_i\right| = \prod_{i=1}^{n} E|X_i| = (E|X|)^n < \infty$

$$E[Z_n \mid Z_{n-1} \dots Z_1] = E[X_n Z_{n-1} \mid Z_{n-1}, \dots, Z_1]$$

$$= Z_{n-1} E[X_n \mid Z_{n-1}, \dots, Z_1]$$

X_n is independent of $Z_{n-1} \dots Z_1$

$$\overset{\uparrow}{=} Z_{n-1} \cdot E[X_n]$$

$$= Z_{n-1}$$

Reason: X, Y are discrete r.v.

$$E[XY \mid Y=y]$$

$$= \sum_{k \in Range(X)} k \, P(XY = k \mid Y=y)$$

$$\overset{y\neq 0}{=} \sum_{k \in Range(X)} k \, P\left(X = \frac{k}{y} \,\middle|\, Y=y\right)$$

$$= \sum_{u \in Range(X)} uy \, P(X=u \mid Y=y)$$

$$= y \, E[X \mid Y=y]$$

$$\Rightarrow E[XY \mid Y] = Y \, E[X \mid Y]$$

2(a) $X_i = \begin{cases} 2 & \text{wp } \frac{1}{2} \\ 0 & \text{wp } \frac{1}{2} \end{cases}$ iid.

$$Z_n = \prod_{i=1}^{n} X_i$$

(Product martingale)

$P(Z_n = 0) = 1 - \frac{1}{2^n}$; $E[Z_n] = 1$

$P(Z_n = 2^n) = \frac{1}{2^n}$

Key results:- $\cdot$ $Z_n \to 0$ in Probability Ex($Z_n \to 0$ wp 1) as $n \to \infty$

$\cdot$ $E[Z_n] \to 1$

2 (b) Exponential Martingale

X_i iid $\sim X = \begin{cases} 1 & \text{wp } \frac{1}{2} \\ -1 & \text{wp } \frac{1}{2} \end{cases}$ $\quad S_n = \sum_{i=1}^{n} X_i \quad S_0 = 0$

$i \geq 1$

let $g : (-\infty, \infty) \to \mathbb{R}$

$r \in \mathbb{R}$, $E[e^{rX}] = \dfrac{e^{r} + e^{-r}}{2}$ and $g(r) := \ln(E[e^{rX}])$

Fix $t \in \mathbb{R}$

$$Z_n = e^{tS_n - ng(t)} = \prod_{i=1}^{n} \underbrace{e^{tX_i - g(t)}}_{Y_i}$$

$$\Rightarrow Z_n = \prod_{i=1}^{n} Y_i \qquad E|Y_i| = 1 \qquad Y_i \text{ are iid}$$

$\{Z_n\}_{n \geq 1}$ - martingale

lemma 3.2

let $\{Z_n : n \geq 1\}$ be a martingale

$\forall\ n \geq i \geq 1$

$$E[Z_n \mid Z_i, Z_{i-1}, \dots, Z_1] = Z_i$$

Proof :- $i = n-1$, - follows from definition of martingale

Tower Property :

X, Y, Z - discrete random variables

$$E[E[X \mid Y, Z] \mid Y] = E[X \mid Y]$$

Proof of Tower Property :-

$y \in Range(Y)$ $\quad z \in Range(Z)$

- $E[X \mid Y=y, Z=z] = \sum_{k \in Range(X)} k \, P(X=k \mid Y=y, Z=z) := h(y,z)$

- $E[h(Y,Z) \mid Y=y] = \sum_{\substack{m \in Range(Y) \\ t \in Range(Z)}} h(m,t) \, \mathbb{P}(Y=m, Z=t \mid Y=y)$

Property of Conditional Expectation

$$= \sum_{t \in Range(Z)} h(y,t) \, \mathbb{P}(Z=t \mid Y=y)$$

$$= \sum_{t \in Range(Z)} \left[\sum_{k \in Range(X)} k \, P(X=k \mid Y=y, Z=t) \right] \mathbb{P}(Z=t \mid Y=y)$$

$$= \sum_{t \in Range(Z)} \sum_{k \in Range(X)} k \, P(X=k \mid Y=y, Z=t) \, \mathbb{P}(Z=t \mid Y=y)$$

$$= \sum_{t \in Range(Z)} \sum_{k \in Range(X)} k \, \frac{\mathbb{P}(X=k, Y=y, Z=t)}{\mathbb{P}(Y=y)}$$

Rearrangement okay as non-negative series

$$= \sum_{k \in Range(X)} k \, \frac{1}{\mathbb{P}(Y=y)} \sum_{t \in Range(Z)} \mathbb{P}(X=k, Y=y, Z=t)$$

$$= \sum_{k \in Range(X)} k \, \frac{1}{\mathbb{P}(Y=y)} \mathbb{P}(X=k, Y=y) = E[X \mid Y=y]$$

$$\Rightarrow E\big[E[X \mid Y, Z] \,\big|\, Y \big] = E[X \mid Y]$$

Verify $n = i+1$

$$\mathbb{E}[Z_n \mid Z_i, Z_{i-1}, \dots Z_1]$$

$$= \mathbb{E}[Z_{i+1} \mid Z_i, Z_{i-1}, \dots Z_1] = Z_i$$

by (1) of Martingale Property

Assume $n = i+k$

$$\mathbb{E}[Z_{i+k} \mid Z_i, Z_{i-1}, \dots Z_1] = Z_i$$

Prove: $n = i+k+1$

$$\mathbb{E}[Z_{i+k+1} \mid Z_i, Z_{i-1}, \dots, Z_1]$$

Tower Property generalized

$$= \mathbb{E}\Big[\, \mathbb{E}[Z_{i+k+1} \mid \underbrace{Z_{i+k}, Z_{i+k-1}, \dots}\, Z_i, \dots Z_1] \,\Big|\, Z_i, Z_{i-1}, \dots Z_1\Big]$$

$\Big(X = Z_{i+k+1}, \; Y = "Z_i, Z_{i-1}, \dots Z_1", \; Z = "Z_{i+k}, Z_{i+k-1}, \dots Z_{i+1}" \Big)$

$$= \mathbb{E}[Z_{i+k} \mid Z_i, Z_{i-1} \dots Z_1]$$

(1) - Martingale Property

Induction hypothesis

$$= Z_i$$

So we have completed the proof by induction. $\square$

Corollary 3.2 :- $E[Z_n] = E[Z_1] \quad \forall n \geq 1$

Proof: lemma 3.2 $\Rightarrow$

$n \geq i \geq 1 \Rightarrow E[Z_n \mid Z_i, Z_{i-1}, \ldots Z_1] = Z_i$

Take $i = 1$ $\quad E[Z_n \mid Z_1] = Z_1$

Take expectation on both sides wrt ?.

$E[E[Z_n \mid Z_1]] = E[Z_1]$

Reason. $E[E[X \mid Y]] = E[X]$

$\Rightarrow$

$E[Z_n] = E[Z_1] \quad \forall n \geq 1$

$\square$

2a HW5 - One typo

- $E[\xi_n \mid \xi_{n-1} = x_{n-1}, \ldots \xi_1 = x_1] = x_{n-1}$

- $E[\xi_n \mid \xi_{n-1}, \ldots, \xi_1] = \xi_{n-1}$ —— (1)