Announcements : – class talk ≡ { – Please start preparing notes
– have a zoom meeting with me to discuss

– Book-keeping, Home work } ⟶ 30% Book-keeping ⟵ $\boxed{\text{Final exam}}$
30% Home work

– Solutions to (Due) H.W / Quizzes ⟶ Course website
⟶ If any of you have handwritten / .. Solutions to other question then do share, they will posted on course website.

– Topical Questions (~ related) ⟶ Post them on course website

Recall :– ✩ – Concrete examples of – Markov chains { Discrete time, Countable state space

– Random walk on weighted graph

✩ – Simple random walk on $\mathbb{Z}$ – Finite length time Horizon

– Simple random walk on $\mathbb{Z}^d$ & infinite length time Horizon.

[Topics] – Next 3 weeks – Discrete time Martingales
(keep in mind ✩)

3 Discrete time Martingales

- Wikipedia on Martingals

Martingale (betting system)

From Wikipedia, the free encyclopedia

For the generalised mathematical concept, see Martingale (probability theory).

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> *Find sources: "Martingale" betting system – news · newspapers · books · scholar · JSTOR (October 2010) (Learn how and when to remove this template message)*

A **martingale** is any of a class of betting strategies that originated from and were popular in 18th-century France. The simplest of these strategies was designed for a game in which the gambler wins the stake if a coin comes up heads and loses if it comes up tails. The strategy had the gambler double the bet after every loss, so that the first win would recover all previous losses plus win a profit equal to the original stake.

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DEFINITIONS martingale

Top Definitions Quizzes Examples British

martingale [mahr-tn-geyl] SHOW IPA

noun

1. Also called **standing martingale.** part of the tack or harness of a horse, consisting of a strap that fastens to the girth, passes between the forelegs and through a loop in the neckstrap or hame, and fastens to the noseband: used to steady or hold down the horse's head.
2. Also called **running martingale.** a similar device that divides at the chest into two branches, each ending in a ring through which the reins pass.
3. *Nautical.* a stay for a jib boom or spike bowsprit.
4. a system of gambling in which the stakes are doubled or otherwise raised after each loss.

Definition 3.1 A sequence of random variables $\{Z_n : n \geq 1\}$ is said to be a Martingale if

$$\mathbb{E}[Z_n \mid Z_{n-1} = z_{n-1}, Z_{n-2} = z_{n-2}, \ldots, Z_1 = z_1] = z_{n-1} \quad -(1)$$

Objectives

- Examples of martingales, ie $\{Z_n\}_{n\geq 1}$ that satisfy $-(1)$ condition $-(1)$
- I.I.D, Markov chain ..... how different / similar is
- Interpretation of $-(1)$

3.1 Examples :-

Example 1 : $\{S_n\}_{n\geq 0}$ simple symmetric random walk on $\mathbb{Z}$.

$S_0 = 0$, $n \geq 1$ $\quad S_n = \sum_{i=1}^{n} X_i \quad X_i \overset{=}{\underset{iid}{}} X \equiv \begin{cases} +1 & \text{w.p } \frac{1}{2} \\ -1 & \text{w.p } \frac{1}{2} \end{cases}$ [Discrete r.v.]

. Let $s_{n-1}, s_{n-2}, \ldots s_1 \in \mathbb{Z}$ such that $\mathbb{P}(S_{n-2} = s_{n-2}, \ldots S_1 = s_1) > 0$

$$E[S_n \mid S_{n-1} = s_{n-1}, S_{n-2} = s_{n-2}, \ldots, S_1 = s_1]$$

$$= \sum_{k \in \mathbb{Z}} k \cdot \mathbb{P}(S_n = k \mid S_{n-1} = s_{n-1}, S_{n-2} = s_{n-2}, \ldots S_1 = s_1)$$

$$= \sum_{k \in \mathbb{Z}} k \cdot \mathbb{P}(S_{n-1} + X_n = k \mid S_{n-1} = s_{n-1}, S_{n-2} = s_{n-2}, \ldots S_1 = s_1)$$

$$= \sum_{k \in \mathbb{Z}} k \cdot \mathbb{P}(X_n = k - s_{n-1} \mid S_{n-1} = s_{n-1}, S_{n-2} = s_{n-2}, \ldots S_1 = s_1)$$

$$= \sum_{k \in \mathbb{Z}} k \cdot \frac{\mathbb{P}(X_n = k - s_{n-1}, S_{n-1} = s_{n-1}, S_{n-2} = s_{n-2}, \ldots S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, S_{n-2} = s_{n-2}, \ldots S_1 = s_1)}$$

$X_n \perp \{S_{n-1}, \ldots S_1\}$

$$\overset{\text{independence}}{=} \sum_{k \in \mathbb{Z}} k \cdot \frac{\mathbb{P}(X_n = k - s_{n-1})\, \mathbb{P}(S_{n-1} = \cancel{s_{n-1}, S_{n-2}} = s_{n-2}, \ldots S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1} \cancel{, S_{n-2}} = s_{n-2}, \ldots S_1 = s_1)}$$

$$= \sum_{k \in \mathbb{Z}} k \cdot \mathbb{P}(X_n = k - s_{n-1})$$

$$= (s_{n-1}+1)\, \mathbb{P}(X_n = 1) + (s_{n-1}-1)\, \mathbb{P}(X_n = -1) + \sum_{k \in \mathbb{Z} / \{s_{n-1}+1,\, s_{n-1}-1\}} k \cdot 0$$

$$= (s_{n-1}+1) \cdot \frac{1}{2} + (s_{n-1}-1)\frac{1}{2} = s_{n-1}$$

As $s_{n-1}, \dots s_1$ were arbitrary $\Rightarrow$

$\{S_n\}_{n \geq 1}$ is a Martingale. $\square$

Example 2 :- $\{X_i\}_{i \geq 1}$ be iid sequence (of discrete random variables) such that

(Product)

$E[X_1] = 1.$

$S = \text{Range}(Z_n)$

$$Z_n = \prod_{i=1}^{n} X_i$$

let $z_{n-1}, z_{n-2}, \dots, z_1 \in \mathbb{R}$ such that

$\mathbb{P}(Z_{n-1} = z_{n-1}, \dots Z_1 = z_1) > 0$

$E[Z_n \mid Z_{n-1} = z_{n-1}, \dots, Z_1 = z_1]$

$$= \sum_{k \in S} k \, \mathbb{P}(Z_n = k \mid Z_{n-1} = z_{n-1}, \dots, Z_1 = z_1]$$

$$= \sum_{k \in S} k \, \mathbb{P}(X_n Z_{n-1} = k \mid Z_{n-1} = z_{n-1}, \dots, Z_1 = z_1)$$

$$= \sum_{k \in S} k \, \mathbb{P}(X_n z_{n-1} = k \mid Z_{n-1} = z_{n-1}, \dots, Z_1 = z_1)$$

$$= \sum_{k \in S} k \, \frac{\mathbb{P}(X_n z_{n-1} = k, Z_{n-1} = z_{n-1}, \dots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \dots, Z_1 = z_1)}$$

independence of $X_n \perp \{Z_{n-1}, \dots, Z_1\}$

$$= \sum_{k \in S} k \, \frac{\mathbb{P}(X_n z_{n-1} = k) \, \cancel{\mathbb{P}(Z_{n-1} = z_{n-1}, \dots, Z_1 = z_1)}}{\cancel{\mathbb{P}(Z_{n-1} = z_{n-1}, \dots, Z_1 = z_1)}}$$

$$= \sum_{k \in S} k \, \mathbb{P}(X_n z_{n-1} = k) \underset{\text{Ex.}}{=} \sum_{u \in S'} u \, z_{n-1} \, \mathbb{P}(X_n = u)$$

$\mathrm{Range}(X_n) = S'$

$$= z_{n-1} \, E[X_n] = z_{n-1}$$

As $z_{n-1}, \dots, z_1$ were arbitrary elements in the range of $Z_{n-1}, \dots, Z_1$ respectively $\Rightarrow$ $\{Z_n\}_{n \geq 1}$ is a Martingale. □

- In Example 2 let us $\{X_i\}_{i \geq 1}$ be i.i.d. $X = \begin{cases} 2 & \text{wp } \frac{1}{2} \\ 0 & \text{wp } \frac{1}{2} \end{cases}$

clearly $E[X] = 1$

$$Z_n = \prod_{i=1}^{n} X_i \quad \text{- is a Martingale}$$

$$\underset{\text{low}}{\mathbb{P}(Z_n = 0)} = \underbrace{1 - \frac{1}{2^n}}_{\text{"high"}}, \quad \underset{\text{high}}{\mathbb{P}(Z_n = 2^n)} = \underbrace{\frac{1}{2^n}}_{\text{"low"}}$$

$$E[Z_n] \underset{\text{mean}}{=} 1$$

$\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots Z_1 = z_1) > 0;\ E[Z_n \mid Z_{n-1} = z_{n-1}, \ldots Z_1 = z_1] = z_{n-1}$

- As $E[Z_n]$ remains a constant, high values will be taken with low probabilities & conversely low values will be taken with high probabilities

- Martingales are different from Markov Chains :-

Markov chain. Each X_n conditioned on the past depends only on X_{n-1}

Martingales Each Z_n: the expected value of Z_n conditioned on its past depends only on Z_{n-1}
- this allows the possibility that Z_n may depend on all $\{Z_i : 1 \leq i \leq n-1\}$

- Even this weak form of conditioning $\Rightarrow$ very nice results about the sequence.

Interpretation of (1)

$z_{n-1}, \quad z_1 \equiv$ Range of $Z_{n-1}, \ldots Z_1$ respectively

$$E[Z_n \mid Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1] = z_{n-1} \quad -(1)$$

Precise Computational view

- $\{Z_i\}_{i \geq 1}$ are discrete r.v.s, understand (1) then (1) well defined

. $P(Z_{n-1} = z_{n-1}, \ldots Z_1 = z_1) > 0$

as the "conventional" conditional expectation

- $\{Z_i\}$ are continuous r.v. with p.d.f, understand (1) well defined

as the "conventional conditional expectation

$f_{Z_{n-1}, \ldots Z_1}(\ldots) \equiv f(z_{n-1}, \ldots z_1) > 0$

. Formulation of $\boxed{\text{Conditional Expectation of } Z_n \text{ given } Z_{n-1}, \ldots Z_1} \rightleftarrows \boxed{Y_{n-1}}$

$$f(z_{n-1}, \ldots, z_1) = E[Z_n \mid Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1] \quad -(*)$$

$$\underbrace{Y_{n-1} := f(Z_{n-1}, \ldots, Z_1)}_{\text{Random variable}}$$

Compute (*) and then obtain expression for $f(z_{n-1}, \ldots, z_1)$ & plug in $Z_{n-1}, \ldots, Z_1$ to obtain Y_{n-1}:

$Y_{n-1} \rightsquigarrow$ Conditional Expectation of Z_n given $Z_{n-1}, \ldots Z_1$

(i) $A = \{\omega \in \Omega \mid Z_1 = z_1, \ldots, Z_{n-1} = z_{n-1}\}$

$$Y_{n-1}(\omega) = E[Z_n \mid Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1] \quad \forall\, \omega \in A$$

(ii) $Y_{n-1} \equiv$ "function of $Z_{n-1}, \ldots Z_1$".

Events concerning Y_{n-1} $\rightsquigarrow$ $\{Y_{n-1} \in B\} \in \mathcal{A}_{n-1}$ – "observable by time $n-1$"

" determinable by events of $(Z_{n-1}, \ldots Z_1)^{-1}(A)$ for suitable "

$\{Z_n\}_{n \geq 1}$ is a martingale then

" $E[Z_n \mid Z_{n-1}, \ldots Z_1]$ " depends only on " Z_{n-1} "

function of $Z_{n-1}, \ldots Z_1$

function of Z_{n-1}

Tuesday class :- – Discrete r.v / Continuous r.v

(HW5) Review Conditional

$\rightsquigarrow$ Book-keeping Expectation