Recall :-

- (Γ, μ) $\Gamma = (V, E)$ of weighted graph
  - μ: weights; V: vertices; E: Edges
- Geometric properties of the graph

- $\{X_n\}_{n \geq 1}$ - reversible Markov chain on V
- Asymptotic Properties of Random walks

Question:- What properties of Graph / Random walk are stable under "minor" perturbation of the graph?

- Volume :- $|B(x,r)|$ ... $|B(x,2r)|$ $r>0$
  Doubling (Think: $\mathbb{R}^d$ ..... $\mathbb{Z}^d$ ... Γ)
- $\boxed{\text{Effective Resistance}}$
- Energy form
- Harmonic functions - $\boxed{h(i) = \mathbb{P}_i(T^P < \infty)}$

Poincare
Nash
Isoperimetric

↳ Transition density bounds

... on diagonal bounds ... ... $\leq p^n_{ii} \leq$ ..... Gaussian bounds

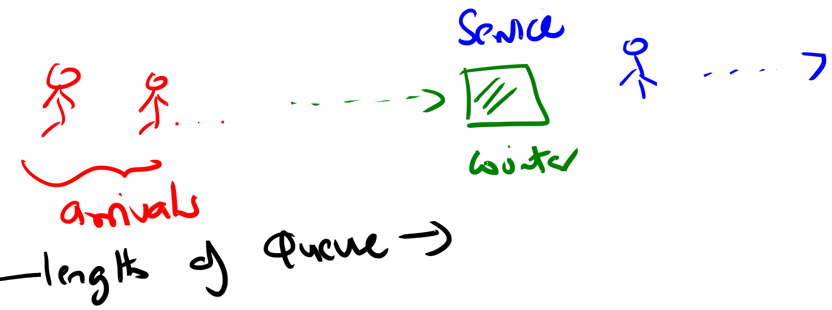
- $\sum_{n=0}^{\infty} p^n_{ii}$ ... Transience / Recurrence
- $\mathbb{P}_i(T^P < \infty) = 1$ / < 1
- 3 technique today

Foster's Criteria ⇠ ..... —— → Lyapunov function

Markov chains — Ordinary differential Equations

Example 6 : [Queueing chain]

- People arrive at service counter [Ticket ; Billing ...]
- Service time — 1 unit of time



let $\xi_i \equiv$ be the # of people joining the queue at time unit i

$\xi_i \equiv$ i.i.d $\xi :=$ $\mathbb{P}(\xi = k) = p_k \quad k = 0, 1, 2, \ldots$

$\left(\sum_{k=0}^{\infty} p_k = 1\right)$

X_n = # of people in the queue at end of time unit n.

$$X_n = \xi_n + \max\{X_{n-1} - 1, 0\} \qquad n \geq 1$$

Ex:- X_n is M.C with a suitable transition matrix P
- irreducible provided $0 < p_0 < 1$; $\exists k > 0$ s.t $p_k > 0$

Q:- Is the chain X_n recurrent or transient ?

Theorem [Foster's Criteria / Lyapunov function]

Let $\{X_n\}_{n\geq 1}$ be a Markov chain with transition matrix P on $V = \mathbb{N} \cup \{0\}$. take $\boxed{f(i) = \mathbb{P}^i(T^0 < \infty)}$

ⓐ the chain is transient $\Longleftarrow \Longrightarrow$

$\exists\, f : V \to \mathbb{R}$ f is bounded - non-constant - such that

$$\sum_{j=0}^{\infty} p_{ij} f(j) = f(i) \quad \forall\, i \neq 0 \quad -(*)$$

$$\boxed{(*) \equiv \Delta f(i) = 0 \quad \forall\, i \neq 0}$$ - harmonic

ⓑ the chain is recurrent $\Longleftarrow \Longrightarrow$ (Ex.)

$\exists\, f : V \to \mathbb{R}$, $\lim_{i \to \infty} f(i) = \infty$ and

$$\sum_{j=0}^{\infty} p_{ij} f(j) \leq f(i) \quad \forall\, i \neq 0 \quad (+)$$

$$\boxed{(+) \equiv \Delta f(i) \leq 0 \quad \forall\, i \neq 0}$$ - Super harmonic

Ex:- In HW 3 ← [Queueing chain]

- $f(i) = i$ works for part ⓑ - $E[\xi] \leq 1$
- find $a_0 \in (0,1)$ works for part ⓐ - $E[\xi] > 1$
  $f(i) = a_0^i$

Differential Equations :

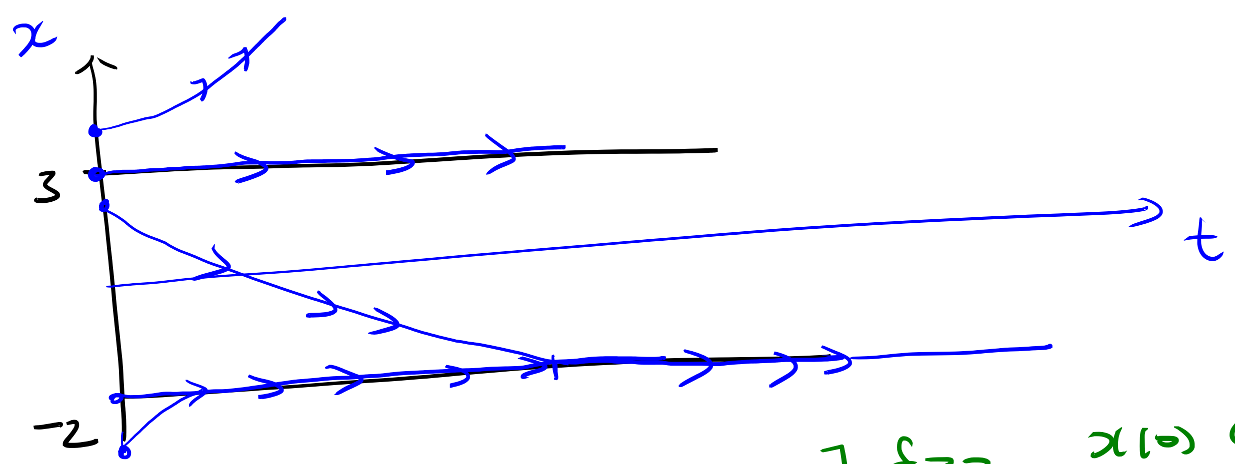
$f: \mathbb{R} \to \mathbb{R}$ $\qquad$ $x: [0,T] \to \mathbb{R}$ $\Big\}$ (*-ode)

- $$\frac{dx}{dt} = f(x)$$

- $a \in \mathbb{R}$, $f(a) = 0$ $\Rightarrow$ if $x(0) = a$ then $x(t) = a$ $\forall\, t > 0$

[Equilibrium points of the (*-ode)]

Example :- $\dfrac{dx}{dt} = (x-3)(x+2)$ $\qquad$ $x(0) = a$



(unstable equilibrium)

(stable equilibrium)

asymptotic stable equilibrium : $\exists\, \delta > 0$ $\quad x(0) \in (-2-\delta, -2+\delta)$ then $x(t) \to -2$ as $t \to \infty$.

Lyapunov-(Condition) function.

Ex: (in higher dimension)
- linear system.
- ? - Lyapunov function -

$V: \mathbb{R} \to [0, \infty)$

- $V(0) = 0$, $\quad V(x) > 0 \quad x \neq 0$
- $\lim_{x \to \infty} V(x) = \infty$
- $\dfrac{d}{dt} V(x(t)) \leq 0 \quad \forall\, t > 0$

then 0 is an asymptotically stable equilibrium for (*-ode).

Example 4: $\{X_n\}_{n\geq 1}$ — Simple random walk on $\mathbb{Z}$.

$$X_0 = 0$$

$$X_n = X_0 + \sum_{i=1}^{n} \xi_i \qquad \xi_i = \text{i.i.d} \quad \xi = \begin{cases} +1 & \text{wp } p \\ -1 & \text{wp } 1-p \end{cases}$$

$$E(\xi_1) = 2p-1 \qquad Var(\xi_1) = 4p(1-p)$$

$p = 1/2$ — recurrent

• $p > \frac{1}{2}$ — $\{X_n\}$ drifts to ∞ } transient

$p < \frac{1}{2}$ — $\{X_n\}$ drifts to $-\infty$

Ex: R-program
Simulate - $\{X_n^p\}_{n\geq 1}$
$p = 1/2$ walk ; $p = \frac{1}{2} + \varepsilon$ walk
- How long do the green & black walks hang around together?

• **Threshold Probabilities**

$a > 0 \qquad \tau^a = \min\{k \geq 1 \mid X_k \geq a\}$

- $\mathbb{P}_0(\tau^a < \infty) = \left(p/_{1-p}\right)^a$
Ex ($p \leq \frac{1}{2}$)

- Distribution of τ^a? - $\begin{cases} E[e^{\lambda \tau^a}] = \cdots \\ \mathbb{P}(\tau^a = n) = \cdots \boxed{?} \\ E[\tau^a] = \cdots \end{cases}$

- **Distribution on X_n** :- n-step

$0 \leq j \leq n$
j: +1 step
$n-j$: -1 step

$$\mathbb{P}(X_n = 2j-n) = \binom{n}{j} p^j (1-p)^{n-j}$$

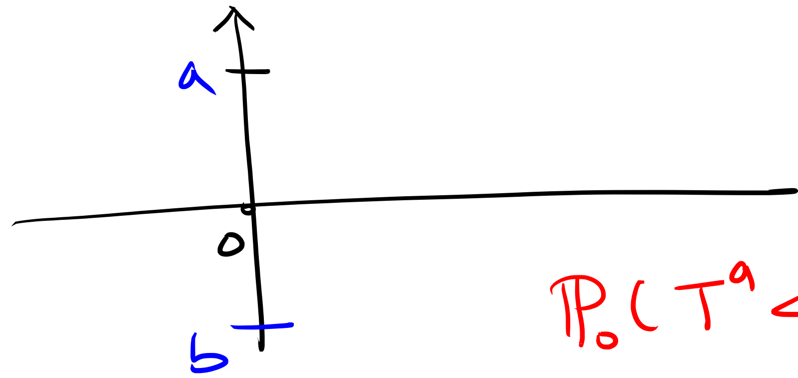
$p = 1/2 \Leftarrow$ Symmetry of walk

- Crossings of $[a,b]$

Distribution of $\frac{T^a}{T^b}$ (CV)

T^a = hitting time of a

T^b = hitting time of b



$\mathbb{P}_0(T^a < T^b) = \dots$ (Gambler's ruin)

- Chernoff bound (Large Deviation Theory)

- [SLLN] $\frac{X_n}{n} \longrightarrow 2p-1$ as $n \to \infty$ wp 1

- [CLT] $P\left(a \le \frac{X - n(2p-1)}{\sqrt{n\,4p(1-p)}} < b\right) \longrightarrow P(N(0,1) \in [a,b])$ as $n \to \infty$

Given $a \in \mathbb{R}$: $\mathbb{P}\left(\frac{S_n}{n} > a\right) \equiv e^{-n I(a)}$ behaviour of I

· $p \neq 1/2$ – Last time to origin of the walk

Abhiti — Markov $\underbrace{\text{Chains}}_{\text{SIR model}}$ in Biology / Branching process

Anitra — Large deviations of X_n

Jainam — Neyman – Pearson lemma $\longleftrightarrow$ Threshold probability of walk

Shaibal — Mixing time of specific models

Ritvik — Liouville Property on $\mathbb{Z}_2$ (++)

Nilya — Random walk in trap models

Akshaya — "ABRACADABRA" — problem

- 45min – 50min class
- 6–7 pages of notes
- 1 – HW problem

$\boxed{\text{Start time}}$

April – 2^{nd} week

- Arrange meeting time with me