Recall:

Non-uniqueness Dirichlet

States: $0 \quad 1 \quad 2 \quad 3$

$\mu_{00} = 1 \qquad \mu_{12} = 1 \qquad \mu_{23} = 1$

$$0 \circlearrowleft 1 \qquad\qquad \begin{tikzcd} 1 \arrow[r, bend left, "1"] & 2 \arrow[l, bend left, "1/2"] \arrow[r, bend left, "1/2"] & 3 \arrow[l, bend left, "1"] \end{tikzcd}$$

$A = \{0\}$

$h(i) = \mathbb{P}^i(T^A < \infty)$

$$\begin{cases} h = 0 & \text{on } A \\ \Delta h = 0 & \text{on } A^c \end{cases}$$

$$\boxed{\begin{array}{lll} \alpha, \beta \geq 0 & f(0) = 1 & \\ & f(1) = \alpha & f(2) = \dfrac{\alpha+\beta}{2} \\ & f(3) = \beta & \end{array}}$$

$\boxed{f \geq h}$ ← Proposition

Example: Simple Random walk on $\mathbb{Z}^d$ $\quad d \geq 1$

$q = 1 - p, \quad 0 < p < 1$

$$\begin{tikzcd} i-1 & i \arrow[l, bend right, "q"'] \arrow[r, bend left, "p"] & i+1 \end{tikzcd}$$

$d = 1$

$X_0 = 0 \qquad \xi_i = \begin{cases} 1 & \text{w.p } p \\ -1 & \text{w.p } 1-p \end{cases}$, independent

$X_n = X_0 + \sum_{i=1}^{n} \xi_i, \qquad X_n \equiv$ m.c on $\mathbb{Z}$

$$\mathbb{P}(X_n = 0 \mid X_0 = 0) = p^n_{00}$$

Criteria:

Recurrence — $\sum_{n=0}^{\infty} p^n_{00} = \infty$

Transience — $\sum_{n=0}^{\infty} p^n_{00} < \infty$

$\boxed{\text{Recall I.S.P}}$

observation :- $p_{00}^{2n+1} = 0$ $\qquad p_{00}^{2n} = \binom{2n}{n} p^n (1-p)^{2n-n}$

$\boxed{d=1}$

Symmetric :- $p = q = 1/2$ [Stirling formula] $p_{00}^{2n} \sim \frac{1}{\sqrt{n}}$

$\Rightarrow \sum_{n=0}^{\infty} p_{00}^n = \infty \Rightarrow$ Recurrent

asymmetric :- $p<q$ or $p>q$ $\quad$ Ex :- $\sum_{n=0}^{\infty} p_{00}^n < \infty \Rightarrow$ Transient

- $X_n = \sum_{i=1}^{n} \xi_i$ $\qquad \xi_i \equiv$ i.i.d $\equiv \begin{cases} 1 & \text{w.p } p \\ -1 & \text{w.p } q = 1-p \end{cases}$

[SLLN] $\qquad \frac{X_n}{n} \xrightarrow{\text{w.p } 1} E[\xi_1] = 1p + (-1)(1-p)$
$= 2p-1$

$p = 1/2$; $2p-1 = 0$, Speed $\equiv 0$; no drift

$p < \frac{1}{2}$, $2p-1 < 0$, "Speed < 0" ; negative drift

$p > \frac{1}{2}$; $2p-1 > 0$, positive drift

- [CLT] $\qquad Var(\xi_1) = E\xi_1^2 - (E\xi_1)^2 = 1 - (2p-1)^2 = 4p(1-p)$

$a < b$
$a, b \in \mathbb{R}$

$$\mathbb{P}\left(a < \frac{X_n - n(2p-1)}{\sqrt{n\, 4p(1-p)}} \le b \right) \xrightarrow[\text{as } n\to\infty]{} \mathbb{P}(N(0,1) \in (a,b])$$

$p = \frac{1}{2}$:
$a = -3$
$b = 3$

$$\mathbb{P}\left(X_n \in (-3\sqrt{n}, 3\sqrt{n}) \right) \xrightarrow[\text{as } n\to\infty]{} \mathbb{P}(N(0,1) \in (-3,3))$$

[Polya –] X_n SRW on $\mathbb{Z}^d$ [Fourier Series]

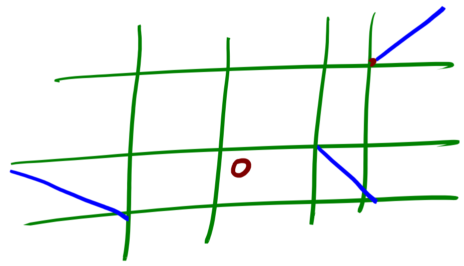
$$\boxed{\sum_{n=0}^{\infty} p^n_{00} \begin{array}{l} < \infty \\ = \infty \end{array} \; ?}$$

Combinatorial approach

- $p = q = 1/2$ Symmetric
  - $d=1$ – recurrent $p^{2n}_{00} \sim \frac{c}{\sqrt{n}}$
  - $d=2$ – recurrent $p^{2n}_{00} \sim \frac{c}{n}$
- $d \geq 3$ – transient [local C.LT] $p^{2n}_{00} \sim \frac{c}{n^{d/2}}$

– Approaches are not robust

Example 4a :– (i) $\tilde{\mathbb{Z}}^2$



on $\mathbb{Z}^2$ at each vertex with Probability 1/2

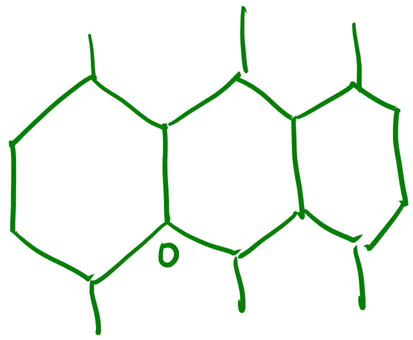
Place a diagonal edge

our guess

X_n SRW on $\tilde{\mathbb{Z}}^2$ will also be reccurrent.

- Alas p^{2n}_{00} is hard to Compute

(ii)



X_n – SRW on $\mathbb{H}$

p^{2n}_{00} is hard to compute

A Simple view of benefits of (P, μ) & structures there of

V – vertex set, $E \subseteq V \times V$, μ – Symmetric weight function, $0 < \mu_{ij} < \infty$

- (P, μ) X_n – R.W from μ $\quad P_{ij} = \frac{\mu_{ij}}{\mu_i}$

  M.C with P on V
  transition matrix

lemma :- $(\mathbb{P}, \mu)$ is a weighted. $\mu(V) < \infty \implies$ Random walk is recurrent

Proof :- [Strategy - Sketch of Proof]

let $p \in V$ and $h : V \to [0,1)$

$$\underbrace{h(i) = \mathbb{P}^{i}(T^{\{p\}} < \infty)}_{\substack{h(p) = 1 \\ \Delta h = 0 \text{ on } V \setminus \{p\}}}$$

Direct Proof Ex.

let $\phi = 1 - h$.

$\phi(p) = 0$

$\Delta \phi = 0$ on $V \setminus \{p\}$

Energy / Dirichlet form

$$\mathcal{E}(\phi, \phi) = \frac{1}{2} \sum_{z, w \in V} \mu_{zw} (\phi(z) - \phi(w))^2$$

[Discrete - Gauss Green Theorem]

$$= -\langle \Delta \phi, \phi \rangle$$

$$= -\sum_{x \in V} \Delta\phi(x)\, \phi(x)\, \mu_x$$

$$= \Delta\phi(p)\, \phi(p)\, \mu_p + \sum_{x \in V \setminus \{p\}} \Delta\phi(x)\, \phi(x)\, \mu_x$$

$$= 0 + 0 = 0$$

. $\phi(\cdot) \equiv$ constant function

As $\phi(p) = 0 \implies \phi(x) = 0 \;\; \forall x \in V \implies h(x) = 1 \;\; \forall x \in V$

In particular $h(i) = \mathbb{P}^{i}(T^{\{p\}} < \infty) = 1 \;\; \forall i \in V$

$\implies \{X_n\}_{n \geq 1}$ is recurrent $\square$

Example 4 $\quad \mathbb{Z}_+ = \{0, 1, \ldots, \}$; $\quad 0 < \alpha$

$$\mu_{n,n+1} = \alpha^n \qquad \underset{n-1}{\bullet} \xrightarrow{\alpha^{n-1}} \underset{n}{\bullet} \xrightarrow{\alpha^n} \underset{n+1}{\bullet}$$

$$p_{n,n+1} = \frac{\mu_{n,n+1}}{\mu_n} = \frac{\alpha^n}{\alpha^n + \alpha^{n-1}} = \frac{\alpha}{1+\alpha} \qquad p_{n,n-1} = \frac{\mu_{n,n-1}}{\mu_n} = \frac{1}{1+\alpha}$$

$\mu(v) < \infty \iff \alpha < 1$

$$\boxed{\frac{\alpha}{1+\alpha} = p \quad \text{ie} \quad p < q \qquad \frac{1}{1+\alpha} = q}$$

$\rightarrow$ right with less probability

$\therefore$ by lemma

X_n on $\mathbb{Z}_+$ is recurrent. $\square$

other graphs :-

① d-array tree

$\underline{d=2}$ $\underline{\text{case}}$: $n \geq 1$ $\quad B_n = \{0,1\}^n$ $\qquad B = \bigcup_{n=1} B_n \cup \{\rho\}$

$B_0 = \{\rho\}$

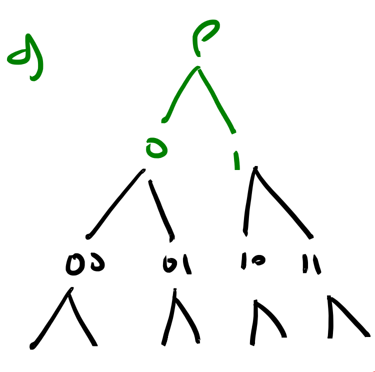
• $x \in B_n \Rightarrow x = (x_1, \ldots x_n)$ define $\alpha(x) = (x_1, \ldots x_{n-1})$
ancestor

$E = \{ \{x, \alpha(x)\} \mid x \in V \; x \notin B_0 \}$

neighbourhood

$\boxed{\text{Fertile graph for computations}}$

Binary tree



$P = (B, E)$

$\Gamma = (V, E)$ $d : V \times V \to [0, \infty) \cup \{\infty\}$

[Metric] $d(x,y) = \begin{cases} \text{length of the shortest path from } x \text{ to } y \text{ in } \Gamma \\ \infty, \text{ if no path exists} \end{cases}$

$r > 0$ $B(x,r) = \{ y \in V \mid d(x,y) \le r \}$

On Binary tree :- $|B(p,n)| = \sum_{k=0}^{n} 2^k$ [volume]

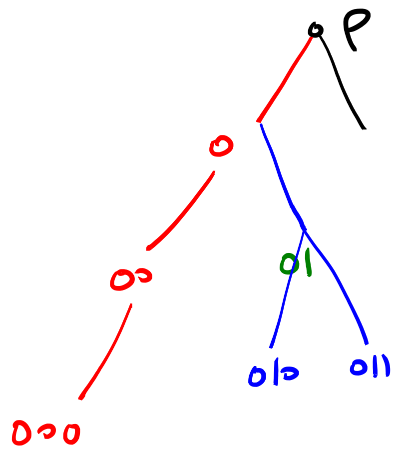
[Poincare - Nash - Isoperimetric] $|\partial B(p,n)| = 2^n$ [Surface area]

Canopy tree :- $(B, E) = \overset{\text{Binary}}{\text{Tree}}$ as defined above

$U = \{ x \in B \mid x = (x_1, \ldots x_n) \quad x_i = 0 \quad \forall\ 1 \le i \le n \text{ for some } n \ge 1 \}$

$f : B \to B$

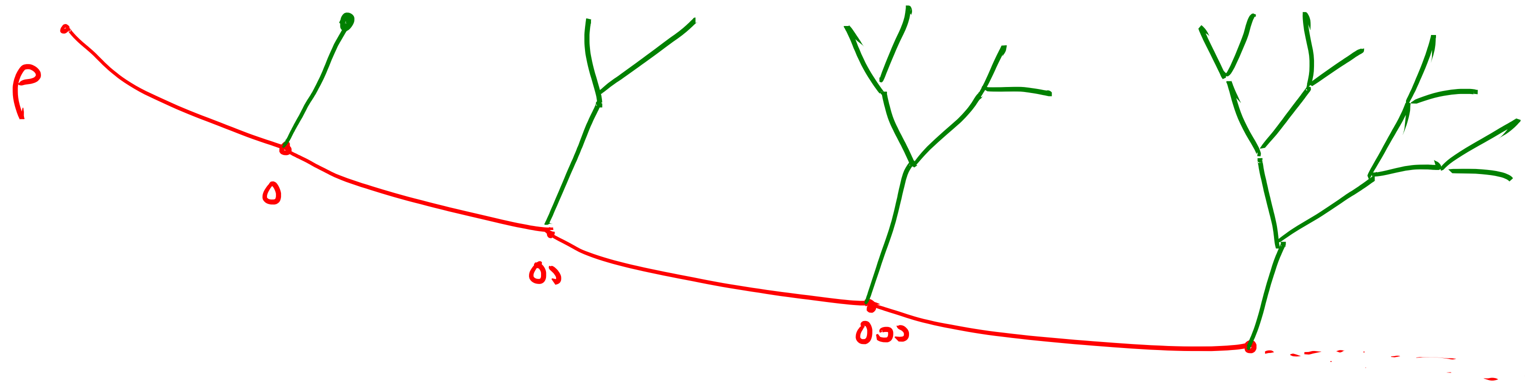
$f(x) =$ point in U that is closest to x



$f(01) = 0$
$f(010) = 0$
$f(011) = 0$
$\vdots$

$$C_{anopy} = \{ x \in B \mid d(x, f(x)) \le d(p, f(x)) \}$$

self avoiding
1-path to ∞



volume grows exponentially