Recall :-

- V - countably infinite / finite vertex set
- $E \subseteq \{ \{i,j\} \mid i,j \in V \}$ - Edge set
- $\underbrace{\Gamma = (V, E)}_{\text{graph}}$

$\mu_{ij} = \mu_{ji}$ Symmetry

$\mu_i = \sum_{j \in V} \mu_{ij}$

$\mu : E \to (0, \infty)$ - weights

$$\mu_{ij} = \begin{cases} \mu(\{i,j\}) & (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

(Γ, μ) weighted graph

$0 < \mu_i < \infty$

- (X_n, P) on S - Markov chain $\quad$ P - transition matrix

$$p_{ij} = \frac{\mu_{ij}}{\mu_i} \qquad i,j \in V$$

- reversible $\quad$, stationary

$\mu_i p_{ij} = \mu_j p_{ji}$ $\qquad$ $\mu_i = \sum_{j \in V} p_{ji} \mu_j$

μ - c'bly additive set function on V

$\mu(\{i\}) = \mu_i \; ; \; \mu(A) = \sum_{i \in V} \mu_i \qquad A \subseteq V$

$\boxed{\mu(V) = 1}$ or $\mu(V) < \infty \Rightarrow \boxed{\pi(A) = \frac{\mu(A)}{\mu(V)}}$

- (X_n, P) - on V - irreducible, aperiodic, π - stationary such that $\exists\; c > 0, \; 0 < \alpha < 1$

$$\|\mathbb{P} \circ X_n^{-1} - \pi\|_{TV} := \sup_{A \in \mathcal{F}} |\mathbb{P} \circ X_n^{-1}(A) - \pi(A)| \leq c\, \alpha^n$$

$\boxed{\text{dependence on } V}$

rate of convergence

"$e^{n \log \alpha}$"

[Convergence to stationarity]

Example 0 :-

$V = \{0,1\}$, $P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$, $\pi(0) = \beta/\alpha+\beta$, $\pi(1) = \frac{\alpha}{\alpha+\beta}$

$$\| P(x_n \in \cdot) - \pi(\cdot) \|_{TV} = \underbrace{|1-\alpha-\beta|}_{\text{eigenvalue of } P}^{n} \left| \mu(\{0\}) - \frac{\beta}{\alpha+\beta} \right|$$

Mixing times :-

- classical – get rates
- modern – $\boxed{\text{large state space}}$ – fix a target distance to reach within the stationary distribution.

Mixing time :- – # of $\underbrace{\text{time steps}}_{\text{depend on } |V|}$ to reach the target distance [sugar in water – mix it]

Statistical Physics $\boxed{\text{Spin Systems}}$ $\xleftrightarrow{\text{interaction}}$ Theoretical Computer Science

- Monte Carlo Simulation : [Dobrushin / Dostika vats]

How long do you run the simulation ?

- sampling
- approximate counting algorithm

"mixing time" : $= O(\log |V|^{\alpha})$
$\alpha > 0$

Mathematics

- Random shuffling of cards
  - Spectral theory
  - number theory
  - Representation theory

$\boxed{\text{Random walk on graphs}}$

Discrete Differential Geometry of graphs

PI / N | LS / Iso inequalities

$\boxed{\text{Discrete Potential Theory}}$
C-Analysis / Differential Equation

Definition of Mixing time

$$d(n) = \max_{i \in V} \| \mathbb{P}^{i} \cdot X_n^{-1} - \pi \|_{TV}$$

$$\boxed{\mathbb{P}^{i}(X_0 = i) = 1}$$

$|V| = \infty$ max-sup

$|V| < \infty$

$$t_{mix} \equiv \text{Mixing time} = \min\{n : d(n) \leq 1/4\}$$

Example 1(b):

$\mathbb{Z}_m = \{0, 1, \dots, m-1\}$ (X_n, P)

$P = [p_{ij}]$

$$p_{ij} = \begin{cases} 1/2 & j = i+1 \bmod(n) \\ \frac{1}{2} & j = i-1 \bmod(n) \\ 0 & \text{otherwise} \end{cases}$$

$\pi \equiv$ uniform on $\mathbb{Z}_m$

- aperiodic $\equiv 2 = d(i) \; \forall i \in V$ ⊗

Lazy (X_n, φ)

$$\varphi = \frac{I + P}{2}$$

$$a_{ij} = \begin{cases} 1/2 & i = j \\ 1/4 & j = (i+1) \bmod(n) \\ 1/4 & j = (i-1 \bmod(n) \\ 0 \end{cases}$$

$$\begin{tikzcd} i-1 & i \arrow[l, bend right, "1/4"'] \arrow[r, bend left, "1/4"] \arrow[loop below, "1/2"] & i+1 \end{tikzcd}$$

Question:- Can we bound mixing time of (X_n, φ) ?

Answer:- $\frac{n^2}{32} \leq t_{mix} \leq n^2$

Example 1(a):

Make reflected random walk $\to$ lazy RRW

$C_2 m^2 \leq t_{mix} \leq C_1 m^2$ [check]

Motivated — largely invisible — Examples
influence

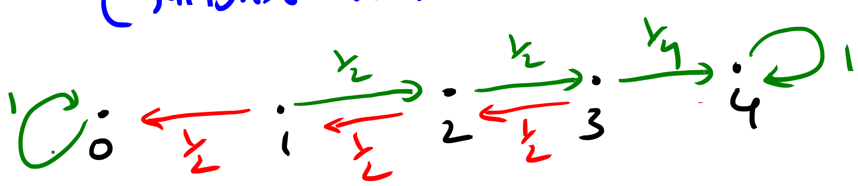
$t_{mix} \longleftrightarrow$ Spectral properties of transition matrix P

Eigen values of P

Example 3 : [Gambler's Ruin chain]

$V = \{0, 1, 2, 3, 4\}$

(X_n, P)



Laxmi	Ganbi
2	2

Toss a coin (fair)

H – Ganbi

T – Laxmi

$X_n =$ Capital of Ganbi

[Hitting times] $T^D = \min\{k \geq 0 \mid X_k \in D\}$

$D = \{4\}$

Question: $\mathbb{P}^2(X_n = 4 \text{ for some } n \geq 1) = \mathbb{P}^2(T^D < \infty)$?

Answer :- $h(i) = \mathbb{P}^i(T^D < \infty)$ $\quad h : V \to [0,1]$

[Markov Property] : $h(0) = 0$, $h(4) = 1$, $h(i) = \frac{1}{2} h(i-1) + \frac{1}{2} h(i+1)$

$i = 1, 2, 3$

$\Downarrow$

$$h(2) = \frac{1}{2}$$

$\square$

Example 3b :

Laxmi & Daughters Casino | Ganbi

4Rs

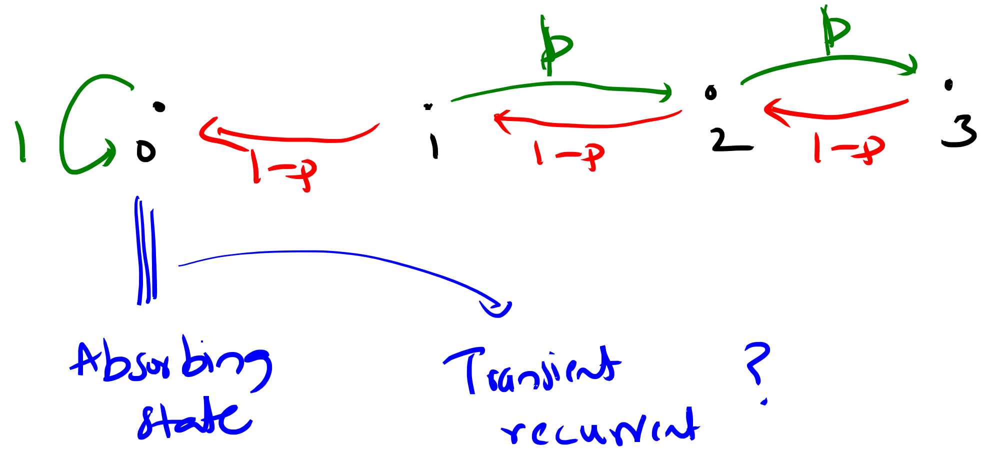
Toss a biased coin w.p p $\quad 0 < p < 1$

H $\Rightarrow$ Ganbi gets 1Rs

T $\to$ Ganbi loses 1Rs

X_n = Capital of Ganbi $\qquad V = \{0,1,2,3,\dots\}$

(X_n, P)



Question

$A = \{0\}$

$\mathbb{P}^2(T^A < \infty) = ?$

$T^A = \min\{k \geq 0 \mid X_k \in A\}$

Answer:

$$h : V \to [0,1]$$

$$h(i) = \mathbb{P}^i(T^A < \infty)$$

$$h(i) = \underbrace{p\,h(i+1) + (1-p)\,h(i-1)}_{\text{[Reccurrence relation using Markov Property]}} \; ; \; i \geq 1$$

- $h(0) = 1$

what is h?

- $1-p > p$ i.e $p < \frac{1}{2} \overset{Ex}{\Longrightarrow} h(i) = 1 \quad \forall i \in V$
- $1-p < p$ i.e $p > \frac{1}{2} \overset{Ex}{\Longrightarrow} h(i) = \left(\frac{p}{1-p}\right)^{i} \quad \forall i \in V / \{0\}$, $h(0) = 1$
- $1-p = p$ i.e $p = \frac{1}{2} \overset{Ex}{\Longrightarrow} h(i) = 1 \quad \forall i \in V$

Proposition [Hitting times]

(X_n, P) is a Markov chain on V

$A \subseteq V \qquad T^A = \min\{k \geq 0 \mid X_k \in A\}$

$h^A : V \to [0,1] \qquad h^A(i) = \mathbb{P}^i(T^A < \infty) \quad - \; (+)$

(a) $h^A(\cdot)$ is a solution to linear system of equations given by

$$h^A(i) = \begin{cases} 1 & i \in A \\ \sum_{j \in V} p_{ij}\, h^A(j) & \text{if } i \notin A \end{cases} \quad - \; (*)$$

(b) If $f : V \to [0, \infty)$ is another solution to $(*)$ then $f(i) \geq h(i) \quad \forall\, i \in V$.

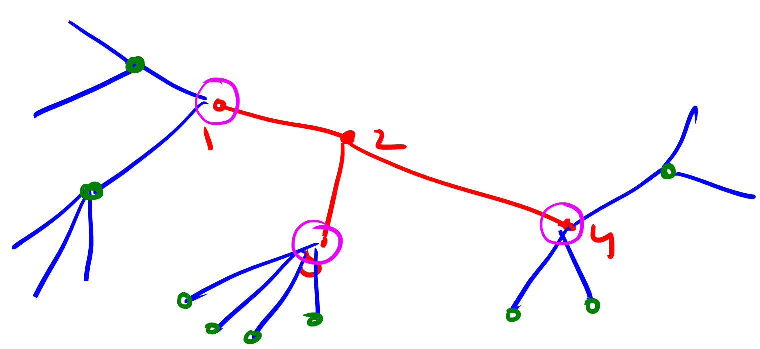
Graph Theoretic Notation :

$i \sim j$ if $\{i,j\} \in E, \qquad \Gamma = (V, E)$

- $A \subseteq V \qquad \partial A = \{ j \in A^c \mid j \sim i \text{ for some } i \in A \}$

"∂A"

$A = \{1,2,3\}$



$\partial_i A = \partial A^c = \{ i \in A \mid i \sim j \quad j \in A^c \}$

Rephrase (*)

$$h^A(i) = \begin{cases} 1 & i \in A \\ \sum_{j \in V} p_{ij}\, h^A(j) & \text{if } i \notin A \end{cases} \quad - (*)$$

is the same as

$h^A(i) = 1$ if $i \in A$, $h^A(i) = \underbrace{P h^A(i)}$ $i \notin A$

$h^A(i) = 1$ if $\in A$, $(P-I)\, h^A(i) = 0$ $i \notin A$

Harmonic

$$\begin{cases} \Delta h^A = 0 & i \notin A \\ h = 1 & i \in A \end{cases}$$

$h^A : V \to [0,1]$ — ...

[Dirichlet Problem]

Solution to Dirichlet Problem <-- ... --> Behaviour of random walk on graph

Next week: – Transience & Recurrence <-- ... --> (P, μ) weighted

[Probability III]

(X_n, P) m.C

[Recurrence (Transience)]