Recall:- → Probability - III - Topics in RWs ← Review

- Examples of Markov chains ····> Martingale Theory

- Student talks - during the course.
  - Set up time in the next two weeks, so that we can agree on a topic via mutual consent.

[Fixed goal - Short two page proof a result]

V - Countable set [finite / c'tbly infinite]

$E = \{ \{i,j\} \mid i,j \in V \}$ - Edge set of the graph.

$\Gamma = (V, E)$.

[weights] $\mu : E \to (0, \infty)$

$\mu(A) = \sum_{i \in A} \mu_i$

- c'tbly additive

If $\mu(V) = 1$ -- Probability

$i,j \in V$: $\mu_{ij} = \begin{cases} \mu(\{i,j\}) & \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$

[Symmetry] $\mu_{ij} = \mu_{ji}$

Assumptions: $i \in V$ $\mu_i = \sum_{j \in V} \mu_{ij} = \sum_{\{i,j\} \in E} \mu(\{i,j\})$

$0 < \mu_i < \infty$

Transition Matrix $P_{V \times V}$

$$p_{ij} = \frac{\mu_{ij}}{\mu_i} \qquad i,j \in V$$

• $p_{ij} \geq 0$, $\sum_{j \in V} p_{ij} = \dfrac{\sum_{j\in V} \mu_{ij}}{\mu_i} = \dfrac{\mu_i}{\mu_i} = 1$

Marker chain : $\Omega = V^{\mathbb{Z}_+} = \{ \omega : \mathbb{Z}_+ \to V \}$, $\mathcal{F}$

$X_n : \Omega \to \mathbb{R}$ $\quad X_n(\omega) = \omega(n)$

$i \in V$, $n \geq 1$ $\quad \mathbb{P}^i (X_0 = i_0, \dots , X_n = i_n) = \underbrace{1_{\{i\}}}_{\text{forces } X_0 = i}^{\{i_0\}} \prod_{k=1}^{n} p_{i_{k-1} i_k}$

$i_k \in V$

[Existence, Unique , ... ← Reading material]

λ is a Probability on V

$X_0 \overset{d}{=} \lambda$, $\mathbb{P}^{\lambda} (X_0 = i_0, \dots X_n = i_n) = \lambda(\{i_0\}) \prod_{k=1}^{n} p_{i_{k-1} i_k}$ — (+)

Ex:- $\mathbb{P} (\underbrace{X_n = j}_{\text{future}} \mid \underbrace{X_{n-1} = i}_{\text{Present}} , \underbrace{X_{n-2} = i_{n-2}, \dots , X_0 = i_0}_{\text{Past}}) = \mathbb{P}(X_n = j \mid X_{n-1} = i)$
$= p_{ij}$

is independent of given present

M.C on (Γ, μ) : [Reversible]

$\mu_i\, p_{ij} = \cancel{\mu_i} \dfrac{\mu_{ij}}{\cancel{\mu_i}} = \dfrac{\mu_{ji}}{\mu_j} \mu_j = \mu_j\, p_{ji}$

$\mu(A) = \sum_{i \in A} \mu_i$

$\mu(V) = 1$ ⊛

Ex:- $\mu_{i_0} \mathbb{P}^{i_0} (X_0 = i_0, \dots , X_n = i_n) = \mu_n \mathbb{P}^{i_n} (X_0 = i_n, \dots , X_n = i_0)$

Assume ⊛ [stationary]

$$\mathbb{P}^{\mu}(X_1=j) = \sum_{i\in V} \mathbb{P}^{\mu}(X_0=i,\ X_1=j) \overset{\oplus}{=} \sum_{i\in V} \mu(\{i\})\, p_{ij}$$

$$= \sum_{i\in V} \cancel{\mu_i} \frac{\mu_{ij}}{\cancel{\mu_i}} = \sum_{i\in V} \mu_{ji} = \mu_j = \mathbb{P}(X_0=j)$$

$$\therefore\ X_0 \overset{d}{=} \mu \implies X_1 \overset{d}{=} \mu$$

Ex : Induction: $X_n \overset{d}{=} \mu \quad \forall n\geq 1$

or otherwise

We have shown:

$$\left.\begin{aligned} &\sum_{i\in V} \mu_i p_{ij} = \mu_j \\ &\mu P = \mu \end{aligned}\right\} \ddagger$$

We assumed ⊛ $\mu(V) = \sum_{i\in V} \mu_i = 1$:

If not: notice always $\ddagger$

$$\pi(A) := \frac{\mu(A)}{\mu(V)}$$

π - will be a probability

$$X_0 \overset{d}{=} \pi \implies X_n \overset{d}{=} \pi \quad \forall n\geq 1$$

Example 1 :- $m \geq 2$.

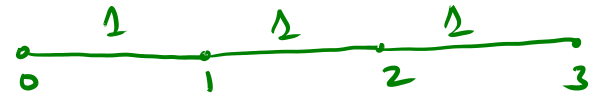
$$E = \{ \{i,j\} \mid |i-j| = 1,\ i,j \in S \}$$

(a) $S = \{0, 1, 2, \dots, m-1\}$

$\{X_n\}_{n \geq 1}$ m.c. on (S,E) $\leftarrow$ $\mu : E \to (0,\infty)$

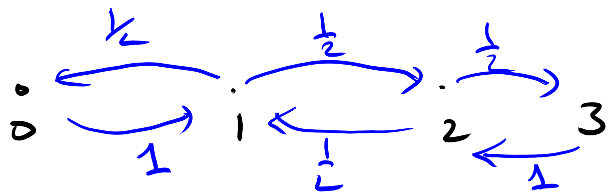
$\mu(\,) \equiv 1$.

$m = 4$:



$\mu(S) = 2m$

$$\mu_i = \begin{cases} 2 & i \neq 0, n-1 \\ 1 & i = 0, n-1 \end{cases}$$



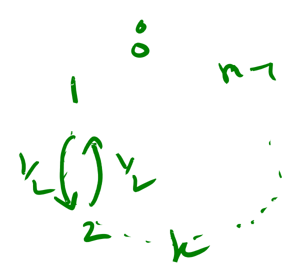
$$p_{ij} = \frac{\mu_{ij}}{\mu_i}$$

X_n reflected simple random walk

- π - stationary distribution ... comes from μ -- $\pi(\{i\}) = \begin{cases} \frac{1}{2m} & i = 0, n-1 \\ \frac{2}{2m} & i \neq 0, n-1 \end{cases}$

$\boxed{\text{Ex :-. irreducible, periodicity} \equiv 2;\ \text{Recurrent states} \equiv S.}$

(b) $S = \mathbb{Z}_m$ $\quad E' = \{0, m-1\} \cup E$



$$p_{ij} = \begin{cases} \frac{1}{2} & i = j+1 \bmod (m) \\ \frac{1}{2} & i = j-1 \bmod (m) \\ 0 & \text{otherwise} \end{cases}$$

$\{X_n\}_{n \geq 1}$ m.c with transition matrix P $\quad \pi$ - uniform(S)

Random walk on m-cycle

Theorem – LIMIT :- $(\{X_n\}_{n\geq 1}, P)$ is a Markov chain, S – finite / cntble infinite

· irreducible, stationary distribution π, aperiodic

$$P^n = [p^n_{ij}] = [\mathbb{P}(X_n = j \mid X_0 = i)]$$

$$\forall j \in S \qquad \lim_{n\to\infty} p^n_{ij} = \pi(\{j\})$$

Applications :- Convergence rate to stationarity $|p^n_{ij} - \pi(\{j\})| = o(\cdot)$

Distance between two Probabilities

$(S, \mathcal{F}, \mathbb{P})$ and $(S, \mathcal{F}, \mathbb{Q})$ two Probability space

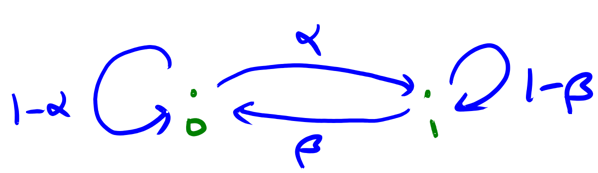
Total variation distance $\equiv \|P - Q\|_{TV} := \max_{A \in \mathcal{F}} |\mathbb{P}(A) - \mathbb{Q}(A)|$

Convergence rate : $|\mathbb{P}(X_n \in A) - \pi(A)| \longrightarrow 0$ at what rate ?

$|S| < \infty$ understand $\|P(X_n \in \cdot) - \pi(\cdot)\|_{TV} \longrightarrow 0$?

Example 0 :- $S = \{0, 1\}$ $\quad 0 < \alpha, \beta < 1$

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$



$\{X_n\}_{n\geq 1}$ M.C with transition matrix P on S

Stationary π $\cdot\ \pi(\{0\}) = \beta/\alpha+\beta$ $\qquad \pi(\{1\}) = \frac{\alpha}{\alpha+\beta}$ [Ex.] — XX

$\pi P = \pi$

Theorem - LIMIT: $\quad X_n$ - aperiodic, irreducible, stationary

- $X_n \xrightarrow{d} \pi$ as $n \to \infty$

- $p_{ij}^n \to \pi(\{j\})$ as $n \to \infty$

- $\| \mathbb{P}(X_n \in \cdot) - \pi(\cdot) \|_{TV} \to 0$ as $n \to \infty$

$(\because S = \{0,1\},\ \mathcal{F} = \{\phi, S, \{0\}, \{1\}\})$

Ex:- $X_0 \overset{d}{=} \mu$ $\quad$ f

$$\mathbb{P}(X_n = 0) = \beta/\alpha+\beta + (1-\alpha-\beta)^n \left(\mu(\{0\}) - \beta/\alpha+\beta\right)$$

$$\mathbb{P}(X_n = 1) = \frac{\alpha}{\alpha+\beta} + (1-\alpha-\beta)^n \left(\mu(\{1\}) - \frac{\alpha}{\alpha+\beta}\right)$$

$$\| P(X_n \in \cdot) - \pi(\cdot) \|_{TV} = \max_{A \in \mathcal{F}} | \mathbb{P}(X_n \in A) - \pi(A) |$$

$$\underset{\mathcal{F}}{\overset{\text{structure}}{=}} \max\{ |\mathbb{P}(X_n = 0) - \pi(\{0\})|, |\mathbb{P}(X_n = 1) - \pi(\{1\})| \}$$

$$\underset{=}{f, XX} \max\{ |(1-\alpha-\beta)^n (\mu(\{0\}) - (\beta/\alpha+\beta))|, |(1-\alpha-\beta)^n (\mu(\{1\}) - \frac{\alpha}{\alpha+\beta})| \}$$

$\mu(\{0\}) + \mu(\{1\}) = 1$ $\qquad = |(1-\alpha-\beta)|^n \cdot |(\mu\{0\} - \beta/\alpha+\beta)|$

$\Downarrow$

$$\boxed{\text{Convergence to stationary is exponential as } 0 < \alpha+\beta < 2}$$

Theorem - Convergence: $|S| < \infty$; $\{X_n\}_{n \geq 1}$ P irreducible, aperiodic, stationary

$\exists$ $C > 0$ & $0 < \eta < 1$:

$$\sup_{\mu} \| \mathbb{P} \cdot X_n^{-1} - \pi \|_{TV} \leq C \eta^n$$

μ - Probability on S

$X_0 \overset{d}{=} \mu$