Large deviations for random walks with finite M.G.F.

$X_1, X_2, X_3, \ldots$ iid $\mathbb{R}$-valued random variables with

$$E[X_1] = \mu \in \mathbb{R}$$
$$Var(X_1) = \sigma^2 \in (0, \infty)$$

$$S_n := X_1 + X_2 + \cdots + X_n$$

Speed of the walk:

Theorem (i) Strong law of large numbers (SLLN):

$$S_n/n \xrightarrow{a.s} \mu \quad \text{as } n \to \infty$$

i.e. $P\left(\lim_{n\to\infty} S_n/n = \mu\right) = 1$

Second order fluctuations:

Theorem (ii) Central limit theorem (CLT):

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{dist.} N(0,1) \quad \text{as } n \to \infty$$

CLT gives the prob. of S_n deviating from $n\mu$ by an amount $\sqrt{n}$.

$$\{S_n \geq (\mu + a)n\} \qquad a > 0$$

$$P\left(\left|S_n/n - \mu\right| > a\right) \to 0 \quad \text{as } n \to \infty$$

We'd calculate the rate at which it goes to 0.

For large n

$$P(S_n \geq (\mu+a)n)$$
$$\approx P(X_i \geq \mu + a,\ 1 \leq i \leq n)$$
$$= \{P(X_i \geq \mu + a)\}^n$$

$$= \left(f(a)\right)^n$$
$$= e^{n\log(f(a))}$$

Deviations of order $n \to$ Large deviations.

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Theorem (iii)

Let $\{X_i\}_{i\geq 1}$ be i.i.d. random variables with $P(X_1=0) = P(X_1=1) = 1/2$

Then, for $a > 1/2$

$$\lim_{n\to\infty} \frac{1}{n} \log\left(P\{S_n \geq an\}\right) = -I(a)$$

where $I(z) = \begin{cases} \log 2 + z\log z + (1-z)\log(1-z) & \text{if } z \in [0,1] \\ \infty & \text{otherwise.} \end{cases}$

Proof : If $a > 1$ $P(S_n \geq an) = 0$

$[S_n \leq n]$

$$\frac{1}{n}\log\left(P(S_n \geq an)\right) = -\infty \quad \forall n$$

Now if $1/2 < a \leq 1$

$$P(S_n \geq an) = \sum_{n \geq k \geq an} 2^{-n}\left({}^{n}C_k\right)$$

$$Q_n(a) = \max_{n \geq k \geq an} {}^{n}C_k$$

$$2^{-n} Q_n(a) \leq P(S_n \geq an) \leq 2^{-n}(n+1) Q_n(a)$$

Claim : $\lim_{n\to\infty} \frac{1}{n}\log Q_n(a) = -a\log a - (1-a)\log(1-a)$

$$\frac{1}{n}\left(-n\log 2 + \log Q_n(a)\right) \leq \frac{1}{n}\log P(S_n \geq an) \leq \frac{1}{n}\left(-n\log 2 + \log(n+1) + \log(Q_n(a))\right)$$

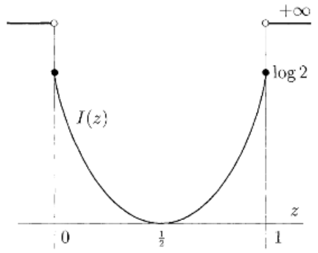
$$\Rightarrow -\log 2 + \frac{1}{n}\log Q_n(a) \leq \frac{1}{n}\log P(S_n \geq an) \leq -\log 2 + \frac{\log(n+1)}{n} + \frac{1}{n}\log(Q_n(a))$$

$$\lim_{n\to\infty} \frac{1}{n} \log P(S_n \geq an) = -\log 2 + \lim_{n\to\infty} \frac{1}{n} \log Q_n(a)$$

$$= -\log 2 - a\log a - (1-a)\log(1-a)$$

Observations

$$I(z) = \begin{cases} \log 2 + z\log z + (1-z)\log(1-z) & \text{if } z \in [0,1] \\ \infty & \text{otherwise.} \end{cases}$$



(i) $P(S_n \leq an)$ $\quad a < 1/2$

$$\lim_{n\to\infty} \frac{1}{n} \log P(S_n \leq an) = -I(a)$$

(ii) SLLN for Ber(1/2) as corollary:

Borel – Cantelli Lemma: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{A_n\}_{n\geq 1}$ is a sequence of events. Say

$$\sum_{k=1}^{\infty} P(A_k) < \infty$$

then the probability that infinitely many of the events A_n occur is 0. i.e.,

$$P\left(\bigcap_{n=1}^{\infty}\left(\bigcup_{k\geq n} A_k\right)\right) = 0$$

Given, $\frac{1}{4} > \delta > 0$ consider $0 < \varepsilon < I(\frac{1}{2}+\delta) < \infty$

choose n_0 such that $\forall n \geq n_0$

$$\left| \frac{1}{n} \log(P(S_n \geq (\tfrac{1}{2}+\delta)n)) + I(\tfrac{1}{2}+\delta) \right| < \varepsilon$$

$$\Rightarrow P(S_n \geq (\tfrac{1}{2}+\delta)n) < e^{n(\varepsilon - c)}$$

Similarly choose n_1 so that

$$P(S_n \leq (\tfrac{1}{2}-\delta)n) < e^{n(\varepsilon - c)}$$

$$\sum_{n=1}^{\infty} P\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| \geq \delta\right)$$

$$\leq \sum_{n=1}^{N} P\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| \geq \delta\right) + \sum_{n=N+1}^{\infty} 2e^{n(\varepsilon - c)} < \infty$$

$[N = \max\{n_0, n_1\}]$

$$A_n^{\delta} : \left\{ \left|\frac{S_n}{n} - \frac{1}{2}\right| \geq \delta \right\}$$

$$P\left(\bigcap_{n=1}^{\infty} \left(\bigcup_{k \geq n} A_k^{\delta}\right)\right) = 0$$

$$P\left(\bigcup_{m=1}^{\infty} \left(\bigcap_{n=1}^{\infty} \left(\bigcup_{k \geq n} A_k^{\frac{1}{m}}\right)\right)\right) = 0$$

$$\left\{\lim_{n\to\infty} \frac{S_n}{n} = \frac{1}{2}\right\} = \left(\bigcup_{m=1}^{\infty} \left(\bigcap_{n=1}^{\infty} \left(\bigcup_{k \geq n} A_k^{1/m}\right)\right)\right)^c$$

$$\Rightarrow P\left(\lim_{n\to\infty} \frac{S_n}{n} = \frac{1}{2}\right) = 1.$$

A generalization to random walks with finite M.G.F.

Theorem (IV): Let $\{X_i\}_{i \geq 1}$ be a sequence of i.i.d. random variables with $\phi(t) = E(e^{tX_1}) < \infty \ \forall t \in \mathbb{R}$

Define: $S_n = X_1 + X_2 + \cdots + X_n$, $n \in \mathbb{N}$

Then for $a > E[X_1]$

$$\lim_{n\to\infty} \frac{1}{n} \log P(S_n \geq an) = -I(a)$$

where $I(z) = \sup_{t \in \mathbb{R}} (zt - \log(\phi(t)))$

Proof: WLOG take $a = 0$ and $E[X_1] < 0$

$\tilde{X}_i = X_i - a$

$\tilde{\phi}(t) = e^{-at}\phi(t)$

$\Rightarrow \tilde{I}(0) = I(a)$

Degenerate r.v. case is trivial.

So consider X_i's to be non-degenerate.

Now say $\rho = \inf_{t \in \mathbb{R}} \phi(t)$ and note that

$I(0) = -\log \rho$ [take $I(0) = \infty$ if $\rho = 0$.]

We'd have to prove that $\lim_{n\to\infty} \frac{1}{n} \log\{P(S_n \geq 0)\} = \log(\rho)$

Say $F(x) = P(X_1 \leq x)$ be the cdf of X_1

then $\phi'(t) = \int_{-\infty}^{\infty} x e^{tx}\, dF(x)$

and $\phi''(t) = \int_{-\infty}^{\infty} x^2 e^{tx}\, dF(x)$ [$dF(x)$ is such that we'd get appropriate expressions for discrete and continuous random variables]

Note that $\phi''(t) \geq 0$ [$\because X_1$ is non-degenerate] and so $\phi(t)$ is strictly convex.

(i) $P(X_1 < 0) = 1$, then $\phi'(t) < 0 \quad \forall t$ and hence $\phi(t)$ is strictly decreasing and $\lim_{t\to\infty} \phi(t) = \rho = 0$ and in that case $P(S_n \geq 0) = 0$

$\phi(t) = \int_{\mathbb{R}} e^{tx}\, dF(x)$

$\frac{1}{n}\log\{P(S_n \geq 0)\} = -\infty \quad \forall n$

$\Rightarrow \lim_{n\to\infty} \frac{1}{n}\log\{P(S_n \geq 0)\} = \log \rho = -\infty.$

(ii) If $P(X_1 \leq 0) = 1$ and $P(X_1 = 0) > 0$, then again $\phi'(t) < 0 \quad \forall t$ as X_1 is non-degenerate and $\lim_{t\to\infty} \phi(t) = \rho = P(X_1 = 0)$ and then,

$P(S_n \geq 0) = P(X_1 = X_2 = \cdots = X_n = 0) = \rho^n$

$\Rightarrow \lim_{n\to\infty} \frac{1}{n}\log\{P(S_n \geq 0)\} = \log \rho$

(iii) If $P(X_1 < 0) > 0$ and $P(X_1 > 0) > 0$ then, $\lim_{t\to\pm\infty} \phi(t) = \infty$

and as $\phi(t)$ is strictly convex, $\exists$ a unique $\tau > 0$ with $\phi'(\tau) = 0$ and $\phi(\tau) = \rho$

[$\tau > 0$ as $\phi'(t) < 0 \quad \forall t < 0$]

Hence, $P(S_n \geq 0) = P(e^{\tau S_n} \geq 1) \overset{(*)}{\leq} \dfrac{E(e^{\tau S_n})}{1}$

$= E(e^{\tau(\sum_{i=1}^{n} X_i)}) = (\phi(\tau))^n = \rho^n$

[(*) By Chebyshev's inequality.]

$\Rightarrow \limsup_{n\to\infty} \frac{1}{n}\log\{P(S_n \geq 0)\} \leq \log \rho$

To prove the other inequality, consider the i.i.d. sequence $\{\hat{X}_i\}_{i\geq 1}$ of random variables with c.d.f. given by

$\hat{F}(x) = \frac{1}{\rho} \int_{(-\infty, x]} e^{\tau y}\, dF(y)$

$\hat{F}$ is called the Cramer-transform of F.

Define $\hat{S}_n := \hat{X}_1 + \hat{X}_2 + \cdots + \hat{X}_n$.

$P(S_n \geq 0) = \rho^n E(e^{-\tau \hat{S}_n} 1_{\{\hat{S}_n \geq 0\}})$

$\liminf_{n\to\infty} \frac{1}{n}\log E(e^{-\tau \hat{S}_n} 1_{\{\hat{S}_n \geq 0\}}) \geq 0$

So, $\liminf_{n\to\infty} \frac{1}{n}\log\{P(S_n \geq 0)\} \geq \log \rho$

$\lim_{n\to\infty} \frac{1}{n}\log P(S_n \geq 0) = \log \rho$