Recall :-

Theorem 3.12 : • $X_1, X_2, \ldots X_n$, i.i.d X with $\sum_{i=1}^{n} X_i = S_n$

• $\exists\, \delta > 0$ $r \in (-\delta, \delta)$, $Ee^{rX} < \infty$

• J be a stopping time wrt $\{X_n\}_{n\geq 1}$

$$E\left[e^{rS_J - J\eta(r)}\right] = 1 \quad \text{where} \quad r \in (-\delta, \delta)$$

and $\eta(r) = \log\left(Ee^{rX}\right)$.

Proof :- • [Optional Sampling Theorem] : $Z_n = e^{rS_n - n\eta(r)}$

under $|\text{Range}(X)| < \infty$

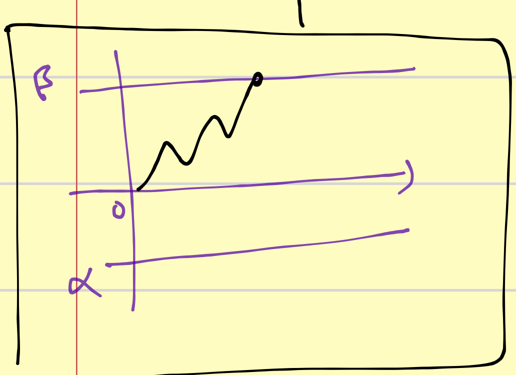
Lemma 3.13 let $\{X_n\}_{n\geq 1}$ i.i.d X & $X \not\equiv 0$

For each $n \geq 1$, $S_n = \sum_{i=1}^{n} X_i$

$\beta < 0 < \alpha$

Threshold stopping times

$$J = \min\{n \geq 1 \mid S_n \geq \alpha \text{ or } S_n \leq \beta\}$$



$$\Rightarrow P(J < \infty) = 1 \quad \& \quad E[J^m] < \infty \quad \forall m \geq 1$$

Theorem 3.14 : let $\{X_n : n \geq 1\}$ be a sequence (of Discrete) IID rvs X. For $n \geq 1$ let $S_n = \sum_{i=1}^{n} X_i$. Assume :-

(M1) – $E[X] < \infty$.

(S1) – J be a stopping time for $\{X_n\}_{n\geq 1}$ & $E(J) < \infty$

then $E(S_J) = E[X]\, E[J]$.

Proof of Theorem 3.14 :-

(Discrete Fubini) Result from Analysis :- $\{a_{jk}\}_{j\geq 1,\, k\geq 1}$ $\quad a_{jk} \in \mathbb{R}$

If $\displaystyle\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |a_{jk}| < \infty$ then $\displaystyle\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{jk}$

As $E(J) < \infty \implies P(J<\infty) = 1$

$\therefore$ $S_J \equiv$ is well defined.

$$E\left[|S_J|\right] = \sum_{k\in R_S} |k|\, \mathbb{P}(S_J = k) \quad \text{where}$$

$R_J \equiv$ countable set $\supseteq \bigcup_{n=1}^{\infty} \mathrm{Range}(S_n)$

Ex: Requires applying on finite sums & limits

$$= \sum_{k\in R_J} |k| \sum_{n=1}^{\infty} \mathbb{P}(S_J = k,\ J=n)$$

[Rearrangement non-negative]

$$= \sum_{n=1}^{\infty} \sum_{k\in R_J} |k|\, \mathbb{P}(S_n = k,\ J=n)$$

[Fix $n\geq 1$ and Consider $S_n \mathbb{1}(J=n)$]

$$= \sum_{n=1}^{\infty} E\left(|S_n|\cdot \mathbb{1}(J=n)\right)$$

$$= \sum_{n=1}^{\infty} E\left(\left|\sum_{m=1}^{n} X_m\right| \mathbb{1}(J=n)\right)$$

(Apply Δle inequality on finite sum and take limits)

$$\leq \sum_{n=1}^{\infty}\sum_{m=1}^{n} E\left(|X_m|\mathbb{1}(J=n)\right) \quad (*)$$

[interchange n and m]

Discrete Fubini (order of sums)

$$= \underbrace{\sum_{m=1}^{\infty}} \underbrace{\sum_{n=m}^{\infty}} E\left[|X_m| \, \mathbb{1}(J=n)\right]$$

$$= \sum_{m=1}^{\infty} E\left[\underbrace{|X_m|} \, \underbrace{\mathbb{1}(J \geq m)}\right]$$

By definition: Independent of $X_1, \ldots, X_{n-1}$

$\underbrace{\text{"function of } X_1, \ldots, X_{n-1}\text{"}}$

J stopping time wrt $\{X_n\}_{n\geq 1}$

$[J \geq m] \in \mathcal{A}_{n-1}$

$= (J < n-1)^c$

independence

$$\overset{\circ}{=} \sum_{m=1}^{\infty} E|X_m| \, P(J \geq m)$$

$\{X_n\}_{n\geq 1}$ i.i.d X

$$= \left[\sum_{m=1}^{\infty} \mathbb{P}(J \geq m)\right] E[|X|]$$

(Discrete Fubini)

$$= E|X| \; E[J] \quad < \infty$$

(m1) and (S1)

Repeat above calculation without $|\cdot|$:-

$$E(S_J) \overset{\circ}{=} \sum_{k \in R_S} k \, \mathbb{P}(S_n = k)$$

(inequality at ⊗ $=$)

$$\overset{\circ}{=} \sum_{n=1}^{\infty} \sum_{m=1}^{n} E(X_m, J=n)$$

Reorder of sums

(Discrete Fubini)

$$\overset{\circ}{=} \sum_{m=1}^{\infty} E\left[X_m \sum_{m=n}^{\infty} \mathbb{1}(J=n)\right]$$

$\{X_m\}_{m\geq 1}$ iid X

$$= E[X] \sum_{m=1}^{\infty} P(J \geq m)$$

$$= E[X]\, E[J] \qquad \square$$

- "Strategies for when to stop are not really effective as far as mean is concerned".

Corollary 3.15 :-

Let $\{X_n\}_{n\geq 1}$ iid X & $X \not\equiv 0$

For each $n \geq 1$, $S_n = \sum_{i=1}^{n} X_i$

$\beta < 0 < \alpha$

$$J = \min\{n \geq 1 \mid S_n \geq \alpha \text{ or } S_n \leq \beta\}$$

Threshold stopping times

- $E[X] < 0$
- $\exists\, \delta > 0 \quad E[e^{\delta X}] = 1$

Then $$P(\underbrace{S_J > \alpha}) \leq \exp(-\delta \alpha)$$

"exit (α, β) at α"

Proof :- $\underbrace{\text{Theorem 3.12 (Wald's Identity)}}_{\text{Lemma 3.13}}$

$$\Rightarrow E[J^m] < \infty \quad \forall m \geq 1 \; ; \; E[e^{\delta J}] < \infty$$

$$\therefore \; E[e^{rS_J - J\eta(r)}] = 1$$

$$\Rightarrow \text{At } r = \delta \; ; \; Ee^{\delta X} = 1 \Rightarrow \eta(\delta) = 0$$

$$\Rightarrow E[e^{\delta S_J}] = 1$$

$$\Rightarrow 1 = E[e^{\delta S_J}(S_J \geq \alpha)] + E(e^{\delta S_J} \, S_J \leq \beta)$$

$$\geq \mathbb{P}(S_J \geq \alpha) \underbrace{E(e^{\delta S_J} \mid S_J \geq \alpha)}_{\geq 0} + 0$$

$$\Rightarrow \mathbb{P}(S_J \geq \alpha) \leq \frac{1}{E(e^{\delta S_J} \mid S_J \geq \alpha)}$$

$$\Rightarrow \mathbb{P}(S_J \geq \alpha) \leq e^{-\delta\alpha} \quad \square$$