Solution **Book-Keeping Exercises** :- HW6 #1

- Coupon-collector Problem: • $S = \{1, 2, \dots, n\}$ and

• $X_1, X_2, \dots X_n$ i.i.d Uniform(S) X_i = i^{th} choice of item from S

$k \geq 1$ $\tau_k^n = \inf\{m : |\{X_1, \dots, X_m\}| = k\}$

- (time at which k - distinct coupons have been picked)

Question:- Distribution of $T_n = \tau_n^n$ - (time at which the complete set S has been collected)

Answer:- $\tau_1^n = 1$ Convention $\tau_n^0 = 0$

$1 \leq k \leq n$ $X_{n,k} = \tau_k^n - \tau_{k-1}^n$ - (time taken to get a different coupon from the 1st $k-1$ set)

Ex:- - $X_{n,k} \overset{d}{=} \text{Geometric}\left(\frac{n-(k-1)}{n}\right)$

- $X_{n,k}$ is independent of $X_{n,j}$ $1 \leq j < k$.

- $E(X_{n,k}) = \frac{n}{n-(k-1)}$ $\text{Var}(X_{n,k}) \leq \frac{n^2}{(n-(k-1))^2}$

(Sum of independent Geometric) $T_n = \tau_n^n = \sum_{k=1}^{n} X_{n,k}$

$0 < a_n, b_n$
$a_n \simeq b_n$ $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$

• $E[T_n] = \sum_{k=1}^{n} \frac{n}{n-(k-1)} = n\left(\sum_{m=1}^{n} \frac{1}{m}\right) \simeq n \log n$

• $\text{Var}(T_n) \leq \sum_{k=1}^{n} \frac{n^2}{(n-(k-1))^2} = n^2 \sum_{m=1}^{n} \frac{1}{m^2} \leq c n^2$, for $c = \sum_{m=1}^{\infty} \frac{1}{n^2}$

• $\frac{T_n - n\sum_{m=1}^{n}\frac{1}{m}}{n \log n} = \frac{T_n}{n\log n} - \frac{\sum_{m=1}^{n}\frac{1}{m}}{\log n}$ $\left[\lim_{n\to\infty} \frac{\sum_{m=1}^{n}\frac{1}{m}}{\log n} = 1\right]$

[Chebyshev] $\frac{T_n}{n\log n} - 1 \xrightarrow{p} 0$

• $\mathbb{P}\left(\frac{T_n - n\log n}{n} \leq x\right) \longrightarrow e^{-e^{-x}}$ as $n \to \infty$
$\forall x \in \mathbb{R}$

Recall (March 30^{th})

- Kolmogorov's maximal inequality

$\{Y_n\}_{n\geq 1}$ - non-negative submartingale $\Rightarrow \mathbb{P}\left(\max_{1\leq i\leq m} |Y_i| \geq b\right) \leq \frac{E|Y_m|}{b}$ $\forall n\geq 1, b>0$

Theorem 3.11 (Martingale Convergence Theorem - L^2)

Let $\{Z_n\}_{n\geq 1}$ be a martingale and assume that $\exists M>0$ such that $E[Z_n^2] \leq M$ $\forall n\geq 1$

Then $\exists$ a r.v Z such that

$$\mathbb{P}(Z_n \longrightarrow Z \text{ as } n\to\infty) = 1$$

Proof:- Key steps: Apply Kolmogorov maximal inequality

Fix $k\geq 1$ $\quad Y_n = Z_{k+n} - Z_k$ $\quad \forall n\geq 1$

$Y_n^2 = \dots$

- $\mathbb{P}\left(\max_{1\leq i\leq m} |Z_{k+i} - Z_k| \geq b\right) \leq \frac{E[Z_{k+m}^2] - E[Z_k^2]}{b^2}$

- $m\to\infty$ $\quad E[Z_m^2] \to \alpha$ $\quad$ [increasing sequence bounded above]

$\Rightarrow \forall b>0$ $\quad \lim_{k\to\infty} \mathbb{P}\left(\sup_{i\geq 1} |Z_{i+k} - Z_k| \geq b\right) = 0$ $\quad$ — (1)

Claim: (1) $\Rightarrow$ $\{Z_n\}_{n\geq 1}$ is a Cauchy sequence w.p. 1

$\Rightarrow \exists Z$ r.v st $\mathbb{P}(Z_n \to Z \text{ as } n\to\infty) = 1$

$\square$

Recall ends

Proof of claim :-

$A = \{\omega\in\Omega \mid \forall b>0\ \exists N\geq 1\ |Z_{k+n} - Z_k| < b \quad \forall k\geq N, \forall n\geq 1\}$

$$= \bigcap_{b>0} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \bigcap_{n=1}^{\infty} \{|Z_{k+n} - Z_k| < b\}$$

From (1) : $l\geq 1$; $b = 2^{-l}$ $\quad \exists k_l \geq 1$ st

$$\mathbb{P}\left(\sup_{i\geq 1} |Z_{i+k_l} - Z_{k_l}| > 2^{-l}\right) \leq \frac{1}{2^l}$$

$\left(\begin{array}{c}\text{finite sums and} \\ \text{take limits}\end{array}\right) \Rightarrow \quad \sum_{l=1}^{\infty} \mathbb{P}\left(\sup_{i \geq 1} |Z_{i+k_l} - Z_{k_l}| > 2^{-l} \right) < \infty$

If $\quad A_l = \left\{ \sup_{i \geq 1} |Z_{i+k_l} - Z_{k_l}| > 2^{-l} \right\}$

$$\Rightarrow \sum_{l=1}^{\infty} \mathbb{P}(A_l) < \infty.$$

So by Borel-Cantelli lemma

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \right) = 0$$

$$\Rightarrow \quad \mathbb{P}(A_l \text{ occur i.o.}) = 0$$

$$\Rightarrow \mathbb{P}(\{\omega \in \Omega \mid \omega \in A_l \text{ for infinitely many } l \in \mathbb{N}\}) = 0$$

$$\Rightarrow \mathbb{P}(\{\omega \in \Omega \mid \omega \in A_l \text{ for finitely many } l \in \mathbb{N}\}) = 1$$

$$\Rightarrow \mathbb{P}\Big(\{\omega \in \Omega \mid \underbrace{\exists l_0 \ \forall l \geq l_0 \ \sup_{i \geq 1} |Z_{k_l + i} - Z_{k_l}| \leq \tfrac{1}{2^l}}\}\Big) = 1$$

$\Downarrow$ Ex: HW - Book-keeping

$$\mathbb{P}\left(\bigcap_{b>0} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \bigcap_{n=1}^{\infty} \{ |Z_{k+n} - Z_k| < b \} \right) = 1$$

$$\Rightarrow \quad \mathbb{P}(\{Z_k\}_{k \geq 1} \text{ is a cauchy sequence}) = 1 \quad \square$$

Wald's Identities :-

Let $X_1, X_2, \dots, X_n,$ be i.i.d X such that

$\exists \ \delta > 0 \ : \ g(r) = E[e^{rX}] < \infty$ for $r \in (-\delta, \delta)$.

let $\quad K(r) = \log(g(r))$

$$S_n = \sum_{k=1}^{n} X_i$$

Define: $Z_n = \exp(rS_n - nK(r))$

$\{X_i\}_{i\geq 1}$ iid

- $E[Z_n] = E\exp(rS_n - nk(r)) = \prod_{i=1}^{n} E[e^{rX}] \cdot e^{-nk(r)} = 1 < \infty$

- $E[Z_n \mid Z_{n-1} \ldots Z_1] = E\left[Z_{n-1}\left(e^{rX_n - k(r)}\right) \mid Z_{n-1}, \ldots, Z_1\right]$

Z_{n-1} is predictable w.r.t. $\{Z_{n-1} \ldots, Z_1\}$ $\leftarrow$ $= Z_{n-1} e^{-k(r)} E\left[e^{rX_n} \mid Z_{n-1}, \ldots, Z_1\right]$

X_n is independent of $\{Z_{n-1}, Z_1\}$ $\leftarrow$ $= Z_{n-1} e^{-k(r)} E[e^{rX_n}]$

$= Z_{n-1}$

Suppose J is a stopping time w.r.t. A_n, where

A_n - observable event by time n of $\{X_k\}_{k\geq 1}$

Theorem 3.6 (OPTIONAL STOPPING THEOREM)

$E[Z_J] = E[Z_1] \iff E|Z_J| < \infty$ & — $*$

$\lim_{n\to\infty} E(Z_n \mid J > n)\, \mathbb{P}(J > n) = 0.$

If $*$ holds for J (stopping time) and $Z_n = \exp(rS_n - nk(r))$

Then $E\left[e^{rS_J - Jk(r)}\right] = E[Z_1]$

$= E[e^{rS_1 - k(r)}]$

$= E[e^{rX_1}] \cdot e^{-k(r)} = 1.$

Theorem 3.12: If J is stopping w.r.t. A_n as described above and satisfies $*$, then

[Wald's Identity] $E\left[e^{rS_J - Jk(r)}\right] = 1.$

Example 3.12 (a) :-

Let $X_1, X_2, \dots, X_n$, be i.i.d X such that

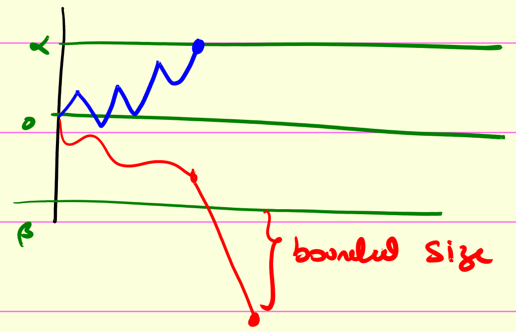
$\exists\ \delta > 0 \ :\ g(r) = E[e^{rX}] < \infty$ for $r \in (-\delta, \delta)$.

let $K(r) = \log(g(r))$

Fix $r \in (-\delta, \delta)$.

and $S_n = \sum_{k=1}^{n} X_i$, $Z_n = \exp(r S_n - n K(r))$

$\beta < 0 < \alpha$, $J = \min\{ n \geq 1 \mid S_n \geq \alpha \text{ or } S_n \leq \beta \}$



When does (*) hold for J ? [ie Hypothesis that is required on X for (*) to hold]

ie

$E|Z_J| < \infty$ & ———— (I)

$\lim_{n \to \infty} E(Z_n \mid J > n)\ \mathbb{P}(J > n) = 0$ ———— (II)

— (*)

For (*) (I)

- If $\mathbb{P}(X \in \{-1, 1\}) = 1$

$\Rightarrow S_J \in \{\alpha, \beta\}$

$\therefore |S_J| \leq |\alpha| + |\beta|$

$\Rightarrow |Z_J| \leq e^{r(|\alpha| + |\beta|)}$

$\Rightarrow E|Z_J| < \infty$.

- If $S = \text{Range}(X)$ $\mathbb{P}(|S| < \infty) = 1$

$\Rightarrow |S_J| < c$ & $E|Z_J| < \infty$.

For ⊛ II : Assume $X \not\equiv 0$ $\beta < 0 < \alpha$

(Ex) $\Rightarrow$ $\exists\, n_0 \in \mathbb{N}$ $\mathbb{P}(S_{n_0} < \beta - \alpha) > 0$
or $\mathbb{P}(S_{n_0} > \alpha - \beta) > 0$ } ensures that S_n exits (β, α) $\forall$ $S_0 \in [\beta, \alpha]$

$\varepsilon = \max\{\mathbb{P}(S_{n_0} < \beta - \alpha), \mathbb{P}(S_{n_0} > \alpha - \beta)\} > 0$

(Ex) $\Rightarrow$ $\mathbb{P}(J > n_0) < 1 - \varepsilon.$

Ex $\Rightarrow$ $\forall\, k \geq 1$ $\mathbb{P}(\underbrace{J \geq n_0 k \mid J \geq n_0(k-1)}_{\text{involves } S_{n_0(k-1)+1}, \dots, S_{n_0 k}}) < 1 - \varepsilon$

$$\Rightarrow \quad \mathbb{P}(J \geq n_0 k) = \prod_{j=1}^{k} \mathbb{P}(J \geq n_0 j \mid J \geq n_0(j-1))$$
$$< (1-\varepsilon)^k = e^{-\alpha k}$$
$$\alpha = |\log(1-\varepsilon)|$$

(+) $\Rightarrow$ • $\mathbb{P}(J < \infty) = 1$ & $E[J^n] = \sum_{k=1}^{\infty} n k^{n-1} \underbrace{\mathbb{P}(J > k)}_{\sim e^{-\alpha k} \text{ for large } k} < \infty.$

⊛ I hold if : Range $(X) \in \{-1, 1\}$ w.p 1
$|\text{Range}(X)| < \infty$

Under this hypothesis $\Rightarrow$ $|Z_n| \mid J > n \leq C$ $\forall n \geq 1$

$\therefore$ $0 \leq E(Z_n \mid J > n)\, \mathbb{P}(J > n) \leq C\, \mathbb{P}(J > n)$

By (+) $\lim_{n\to\infty} E(Z_n \mid J > n)\, \mathbb{P}(J > n) = 0$
and ⊛ II holds

Used to show ⊛ II

Lemma 3.13 : let $\{X_n\}_{n\geq 1}$ be iid $X \not\equiv 0$
For each $n \geq 1$ $S_n = X_1 + X_2 + \dots + X_n$
let $\beta < 0 < \alpha$. $J = \min\{n \geq 1 \; S_n \geq \alpha \text{ or } S_n \leq \beta\}$

$\Rightarrow$ $\mathbb{P}(J < \infty) = 1$ & $E[J^n] < \infty$ $\forall n \geq 1$

Theorem 3.14 [Wald's Identity] Let $\{X_n\}_{n\geq 1}$ be i.i.d. X with $E[X] < \infty$. If J is a stopping time of $\{X_n\}_{n \geq 1}$ & $E[J] < \infty$ then

$$E[S_J] = E[X]\, E[J] \quad \text{where}$$

$S_n = X_1 + \dots + X_n$.

Recall:
- $E[X] = 0 \Rightarrow \{S_n\}_{n\geq 1}$ – martingale

$\otimes$ I
$\otimes$ II
(OST) holds if $E[J] < \infty$ and $|S_{n+1} - S_n| < c$ for some c

Theorem 3.6 $\Rightarrow$ $E[S_J] = E[S_1] = 0 = E[X]\, E[J]$ $\square$

Solution :- 3 HW8

$\times$ $\overset{Y_y^j(\cdot)}{}$ $\left.\begin{matrix}\times\\\times\\\times\\\times\\\circ\end{matrix}\right\}$ $N_y \overset{d}{=} \text{Poisson}(\lambda)$ $\quad \xi(n,x) = \sum_{y\in\mathbb{Z}^d} \mathbb{1}(Y_y^j(n)=x)$

$y \in \mathbb{Z}^d$

$$\sigma^X(n) = E^{\xi}\left[(1-a)^{\sum_{i=0}^{n} \xi(i, X(i))}\right]$$

$$= \exp\left(-\gamma \sum_{y\in\mathbb{Z}^d} \omega^{a,X}(n,y)\right)$$

$$\omega^{a,X}(n,y) = 1 - E_y^Y\left((1-a)^{\sum_{i=0}^{n} \mathbb{1}(Y(i)=X(i))}\right)$$

Proof in
Nitya's Notes : involved a formal calculation.