Probability - Potential theory - Statistical physics

- Random walks
  - short term, long behaviour
  - Hitting Probabilities of sets

"Size" of sets: capacity, Green function, Harmonic function

Electrical networks - effective resistance [Doyle - Snell]

Elliptic Harnack Inequality and Strong Liouville Property

Weighted graphs

V - vertex set - Countable $\qquad \mu$ - weights [locally finite]

$E \subseteq V \times V$ - Edges $\qquad \mu : E \to [0, \infty)$

$\{\mu_{xy} : (x,y) \in V \times V\} \longrightarrow \mu_{xy} := \mu(\{x,y\})$ $\qquad \mu_{xy} \geq 0$, $\mu_{xy} = \mu_{yx}$

$\Gamma = (V, E) \qquad (\Gamma, \mu) \equiv$ weighted graph. $\qquad \left[\mu_x = \sum_{y \sim x} \mu_{xy} > 0\right]$

Transition Probabilities

$$P(x,y) = \frac{\mu_{xy}}{\mu_x} \quad \forall\, x, y \in V$$

Random walk

$\{X_n\}_{n \geq 1}$ to be Markov chain on V with transition matrix P and $X_0 \stackrel{d}{=} Q$. where $P = [P_{xy}]_{x,y \in V}$ $\quad P_{xy} = P(x,y)$

$h : V \to \mathbb{R}$ $\quad$ h is harmonic if $h(x) = \sum_{y \sim x} P_{xy} h(y) \quad \forall x \in V$

$\Leftrightarrow h = Ph \Leftrightarrow (P - I) h = 0 \Leftrightarrow \Delta h = 0$.

Theorem 1: $\{X_n\}_{n \geq 1}$ is recurrent on (Γ, μ)

$\Rightarrow h : V \to [0, \infty)$ is harmonic then h is constant.

Proof:- [sketch] $\{h(X_n)\}_{n \geq 1}$ - Martingale, Convergence results - recurrence to conclude it is constant $\square$

Analysis proof of this [?] $\boxed{\text{the maximum principle}}$

Definition 2 (Elliptic Harnack Inequality):-

(Γ,μ) satisfies Elliptic Harnack Inequality (EHI) if $\exists$ a constant $C_H > 0$ such that

-Given:- $x_0 \in V$, $R \geq 1$, $h: V \to \mathbb{R}$ $\begin{cases} h \geq 0 & \text{in } \overline{B(x_0,2R)} \\ \Delta h = 0 & \text{in } B(x_0,2R) \end{cases}$

Then: $h(x) \leq C_H\, h(y) \quad \forall\, x,y \in B(x_0,R)$ —(EHI)

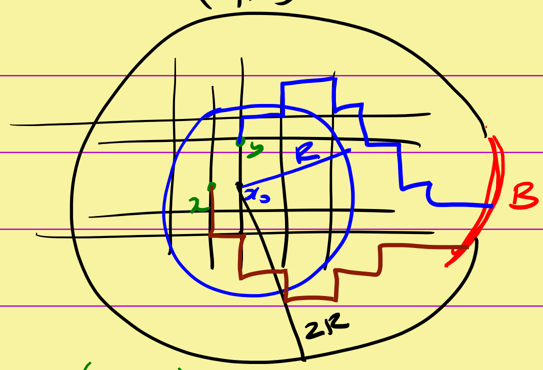
Remarks: $\cdot$ $h \geq 0$ in $B(x_0,2R)$ $\overset{\circledast}{\Longrightarrow}$ $h \geq 0$ in $\overline{B}(x_0,2R)$

[Implication of EHI] $\cdot$ $\max_{x \in B(x_0,2R)} h(x) \leq C_H \min_{y \in B(x_0,2R)} h(y)$

$\cdot$ $\{X_n\}_{n\geq 1}$ random walk on (Γ,μ) $\quad R \geq 1$

$B(x_0,R) := A \subseteq D := B(x_0,2R)$

(Sample)



[Harmonic ≥ 0] $h(x) = \mathbb{P}_x(X_{\tau_D} \in B)$

$\tau_D = \inf\{n \geq 1 \mid X_n \notin D\}$

(EHI) $\quad h(x) \leq C_H\, h(y)$

$\mathbb{P}_x(X_{\tau_D} \in B) \leq C_H\, \mathbb{P}_y(X_{\tau_D} \in B)$

[Ex] $\cdot$ $k > 1$ $\quad \exists\ C_H \equiv C_H(k)$ s.t

$\boxed{R > \frac{2}{k-1}}$ $\quad h \geq 0$ in $\overline{B(x_0,kR)}$ & $\Delta h = 0$ in $B(x_0,kR)$

$h(x) \leq C_H(k)\, h(y) \quad \forall\, x,y \in B(x_0,R)$

Theorem 3 : Suppose (Γ,μ) satisfies EHI

Then it satisfies the strong Liouville Property (SLP)

(i.e $h \geq 0$ harmonic on $V \Rightarrow h$ is constant)

Lemma 3 :- $\mathbb{Z}^d$, with natural weights satisfies EHI.

Proof: – Perhaps – seminal – reading in June middle $\square$

Proof of Theorem 3 :

$h : V \to [0, \infty)$

h is harmonic in V $\quad \Delta h = 0$

Let $x_0 \in V$ and $A_n = B(x_0, 2^n)$. $\quad A_n \subsetneq A_{n+1} \quad \forall n \geq 1$

$\left[\begin{array}{l}\Delta h = 0 \text{ in } A_n \\ h \geq 0 \text{ in } \overline{A_{n+1}}\end{array}\right] \to$ Since (EHI) holds $\forall n \geq 1$ $\exists\, C_H$ s.t.

$$\max_{x \in A_n} h(x) \leq C_H \min_{y \in A_n} h(y) \underset{x_0 \in A_n}{\leq} C_H\, h(x_0)$$

$\Rightarrow$

Since $n \geq 1$ was arbitrary

$$\max_{x \in V} h(x) \leq C_H\, h(x_0) \equiv h \text{ is bounded}$$

Now $\quad \lambda_n = \sup_{x \in A_n} h(x) - \inf_{x \in A_n} h(x) \mathrel{:\equiv} \operatorname{osc}(h; A_n)$

where $\quad \operatorname{osc}(h, B) := \sup_{x \in B} h(x) - \inf_{x \in B} h(x)$.

Assume the following lemma:

Lemma 4 : Suppose (P, μ) satisfies EHI with constant $C_H > 0$. Then

$$\operatorname{osc}(h, B(x_0, R)) \leq (1 - \rho) \operatorname{osc}(h, \overline{B(x_0, 2R)})$$

where $\rho = \frac{1}{2C_H}$.

Using lemma 4 we have

$$\begin{aligned}
\lambda_n = \operatorname{osc}(h, A_n) &\leq (1-\rho) \operatorname{osc}(h, \overline{A_{n+1}}) \\
&= (1-\rho)\left[\sup_{x \in \overline{A_{n+1}}} h(x) - \inf_{x \in \overline{A_{n+1}}} h(x)\right] \\
&\leq (1-\rho)\left[\sup_{x \in A_{n+2}} h(x) - \inf_{x \in A_{n+2}} h(x)\right] \\
&= (1-\rho)\, \lambda_{n+2}
\end{aligned}$$

$\forall n \geq 1$

$osc(h, A_n) \equiv \lambda_n \leq (1-\rho)\, \lambda_{n+2}$ — $(*)$

Iterate, inductively to get

$$0 \leq \lambda_n \leq (1-\rho)^k \lambda_{n+2k}$$

$\left(\because \sup_{x\in V} h(x) \leq C_H h(x_0) \right.$ & $\left. \inf_{x\in V} h(x) \geq 0 \right)$ $\qquad \leq (1-\rho)^k\, C_H\, h(x_0) \qquad \forall n \geq 1\ \forall k \geq 1$

$\left[\dfrac{\rho < 1 \Rightarrow}{\rho \geq 1 \Downarrow \lambda_n = 0} \right]$

let $k \to \infty \Rightarrow \lambda_n = 0 \quad \forall\ n \geq 1$

$\Rightarrow osc(h, A_n) = 0 \quad \forall\ n \geq 1$

$\Rightarrow \sup_{x\in A_n} h(x) = \inf_{x\in A_n} h(x) \quad \forall\ n \geq 1$

$\Rightarrow h(x) = h(x_0) \quad \forall\ x \in V$ $\square$

Proof of lemma 4 :

$A = B(x_0, R) \qquad D = \overline{B}(x_0, 2R)$

WLOG : $\min_{y\in D} h(y) = 0$ & $\max_{y\in D} h(y) = 1$

$[v = \alpha h + \beta]$

- $h(x_0) \geq \frac{1}{2}$

$$\tfrac{1}{2} \leq h(x_0) \leq \max_{x\in A} h(x) \leq C_H \min_{x\in A} h(x) \quad (EHI)$$

$$\Rightarrow h(x) \geq \frac{1}{2C_H} \quad \forall\ x \in A$$

$$osc(h, A) = \sup_{x\in A} h(x) - \inf_{x\in A} h(x)$$

$$\leq \sup_{x\in D} h(x) - \frac{1}{2C_H} \leq \left(1 - \frac{1}{2C_H}\right)(1-0)$$

$$= \left(1 - \frac{1}{2C_H}\right) osc(h, D) \quad \text{— } (\#)$$

- $h(x_0) < \frac{1}{2}$ — take $h' = 1-h$ and repeat the above argument to get $(\#)$ $\square$

Arbitrage pricing Theory for an European Call option

Amul [Stock] let $S_0, S_1, S_2, \ldots$ be the price of Amul stock at time $t=0, t=1, \ldots$

Consider European call option with strike price K and Expiration time m.

(i.e. Confers the right to buy Amul stock at time m for K rupees)

Value / Payoff at Expiration time m $\equiv V_m = (S_m - K)^+$

Q:- What is the arbitrage theory price of the option at time 0 ?

Binomial model :-

$$S_k \begin{cases} \xrightarrow{u} uS_k \\ \xrightarrow{d} dS_k \end{cases} \qquad 0 < d < 1 < u$$

$$S_n = \prod_{i=1}^{n} \xi_i S_0 \qquad \xi_i = \begin{cases} u & \text{wp } p \text{ (Heads)} \\ d & \text{wp } 1-p \equiv q \text{ (Tails)} \end{cases}$$

Assumption:-

- unlimited selling (short) of stock
- unlimited borrow - no Transaction Cost
- "Small investor" - individual moves don't affect market

At time 0

- Sell option at time 0 for V_0
- Buy Δ_0 shares of stock at time 0
- Invests $V_0 - \Delta_0 S_0$ in bank at rate r (<0 or >0)

Wealth at time 1

$$X_1 = \Delta_0 S_1 + (1+r)(V_0 - \Delta_0 S_0)$$

Need at time 1:

$$X_1 = V_1$$

Q:- Find Δ_0, V_0 st $X_1 = V_1$ (regardless of the stock going up or down)

A: $V_1(H) = X_1(H) = \Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0)$

$V_1(T) = X_1(T) = \Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0)$

$\therefore$ $V_1(H) = \Delta_0 S_0 u + (1+r)(V_0 - \Delta_0 S_0)$

$V_1(T) = \Delta_0 S_0 d + (1+r)(V_0 - \Delta_0 S_0)$

$$\Rightarrow \quad \Delta_0 = \frac{V_1(H) - V_1(T)}{(u-d) S_0} = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \text{"}\frac{dV}{dS}\text{"}$$

Δ-hedging

Derivatives

$$V_0 = \frac{1+r-d}{u-d} V_1(H) + \frac{u-(1+r)}{u-d} V_1(T)$$

$d < 1+r < u$

$\equiv$

(Δ-Hedge)

Observations

$\equiv$ independent of p

$\equiv$ $0 \le \tilde{p} = \dfrac{1+r-d}{u-d}$, $0 \le \tilde{q} = \dfrac{u-(1+r)}{u-d}$ $\quad \tilde{p} + \tilde{q} = 1$

$$\boxed{V_0 = \tilde{E}(V_1)}$$

Hw6 - Bk #2

$0 \le \tilde{p} = \frac{1+r-d}{u-d}$, $0 \le \tilde{q} = \frac{u-(1+r)}{u-d}$ $\tilde{p}+\tilde{q}=1$

Risk-neutral Probabilities

$n \in \mathbb{N}$ u>d>r $\Omega = \{H,T\}^n$

$\tilde{\mathbb{P}}(\{\omega_1 \ldots \omega_n\}) = \tilde{p}^{\#\{j:\, \omega_j = H\}} \tilde{q}^{\#\{j:\, \omega_j = T\}}$

$n \ge i \ge 1$. $\xi_i : \Omega \to \{u,d\}$

$\xi_i(\omega) = \omega_i$

Ex $\tilde{E}[\xi_i] = \tilde{p}u + \tilde{q}d$

$= 1+r$

$\forall k \ge 1$ $S_k = \prod_{i=1}^{k} \xi_i \, S_0$

observable events till k-1.

[Hw6 ; #2 Bk] $\tilde{E}[S_k \mid \mathcal{A}_{k-1}] = (1+r) S_{k-1}$

Theorem 1 :- $(1+r)^{-k} S_k$ is a martingale under $\tilde{\mathbb{P}}$ wrt filtration $\mathcal{A}_k$

[Discounted stock price]

Definition 3: A simple European derivative security (SEDS) with expiration time m is a random variable V_m

$\{V_m = c\} \in \mathcal{A}_m$.

Example:- European call $V_m = (S_m - K)^+$

A security is Hedgeable if $\exists$

- $\{X_k\}_{k \ge 0}^{m}$ $\{\Delta_k\}_{k \ge 0}^{m-1}$ st X_0 = initial wealth

$0 \le k \le m-1$; $X_{k+1} = \Delta_k S_{k+1} + (1+r)(X_k - \Delta_k S_k)$ — (*)

and Δ_0 = initial # of stocks bought

(‡) and $X_m = V_m$.

Theorem 4 : [Binomial model is Complete]

Under $\tilde{\mathbb{P}}$ any SEDS is hedgeable ie
$\left[\exists \ X_0, \ \{X_k\}_{k=1}^{n} \ \& \ \{\Delta_k\}_{k=1}^{n-1} \ \text{satisfies} \ \ddagger \right]$

and

$$\ddagger \left\{ \begin{array}{ll} 1 \le k \le n & X_k = (1+r)^{k} \tilde{E}\left[(1+r)^{-n} V_n \,\middle|\, \mathcal{A}_k \right] \\ 0 \le k \le n-1 & \Delta_k = \dfrac{X_{k+1}(\omega_1, \dots, \omega_k ; H) - X_{k+1}(\omega_1 \dots \omega_k ; T)}{S_{k+1}(\omega_1 \dots \omega_k ; H) - S_{k+1}(\omega_1 \dots \omega_k ; T)} \end{array} \right.$$

$$X_0 = \tilde{E}\left((1+r)^{-n} V_n \right)$$

Proof - $\{X_k\}_{k=1}^{n}$ given by $\ddagger$ satisfy $\ddagger$

Suppose X_k satisfies $\ddagger$ then to show

$$X_{k+1} = \Delta_k S_{k+1} + (1+r)\left(X_k - \Delta_k S_k \right)$$

Lemma 5: $\{X_k\}_{k \ge 1}^{n}$ given by $\ddagger$ then under $\tilde{\mathbb{P}}$ $(1+r)^{-k} X_k$ is a martingale.

Proof: Ex

By $\ddagger$ $\quad X_k = (1+r)^{k} \tilde{E}\left[(1+r)^{-n} V_n \,\middle|\, \mathcal{A}_k \right]$

$$\Rightarrow (1+r)^{-k} X_k = \tilde{E}\left((1+r)^{-n} V_n \,\middle|\, \mathcal{A}_k \right]$$

and $(1+r)^{-k} X_k$ is a martingale.

$$(1+r)^{-(k+1)} X_{k+1} = \tilde{E}\left((1+r)^{-n} V_n \mid \mathcal{A}_{k+1}\right)$$

$$(TP) \leftarrow = \tilde{E}\left(\left(\tilde{E}\,(1+r)^{-n} V_n \mid \mathcal{A}_{k+2}\right) \mid \mathcal{A}_{k+1}\right)$$

$$= \tilde{E}\left((1+r)^{-(k+2)} X_{k+2} \mid \mathcal{A}_{k+1}\right)$$

$$X_{k+1} = \tilde{E}\left((1+r)^{-1} X_{k+2} \mid \mathcal{A}_{k+1}\right)$$

$$\Rightarrow \boxed{X_{k+1} = \frac{1}{1+r}\left[\tilde{p}\, X_{k+2}(\omega_1 \dots \omega_k; H) + \tilde{q}\, X_{k+2}(\omega_1 \dots \omega_k; T)\right]}$$

redo algebra

$$\boxed{X_{k+2} = \Delta_{k+1} S_{k+2} + (1+r)\left(X_{k+1} - \Delta_{k+1} S_{k+1}\right)}$$

and

$$\Delta_{k+1} = \frac{X_{k+2}(\omega_1, \dots, \omega_{k+1}; H) - X_{k+2}(\omega_1 \dots \omega_{k+1}; T)}{S_{k+2}(\omega_1 \dots \omega_{k+1}; H) - S_{k+2}(\omega_1 \dots \omega_{k+1}; T)}$$

- Today's class – out of syllabus
- No class on 30^{th} April
- Final Exam – "Quiz" ← 70% – H.W
  ← 30% = (new)

Optional meetings :-

29/4 • Thursday – 6–7 pm
30/4 – Friday 12–1:30 pm
4/5 Tuesday 9–11:30 am
6/5 – Thursday – 6–7 pm
7/5 – Friday 12–1:30 pm

Can Solve Home work Problems

Request a meeting zoom chat

–(May 15^{th} – Final Exam)