Chernoft bounds :-

· Z $\quad Ee^{\gamma Z} < \infty \quad \gamma \in \mathbb{R}$

Markov

- $\mathbb{P}(Z \geq y) \underset{\gamma > 0}{=} \mathbb{P}(e^{\gamma Z} \geq e^{\gamma y}) \leq E[e^{\gamma Z}] \cdot e^{-\gamma y}$

· $X_1, X_2, \dots, X_n$ i.i.d X $\quad E[e^{\gamma X}] < \infty \quad \forall \gamma \in \mathbb{R}$

$$S_n = \sum_{i=1}^{n} X_i$$

$a \in \mathbb{R}$ $\quad \gamma > 0$

$$\mathbb{P}(S_n \geq na) \leq \left[E e^{\gamma S_n}\right] e^{-\gamma n a} = \left[E e^{\gamma X}\right]^n e^{-\gamma n a}$$

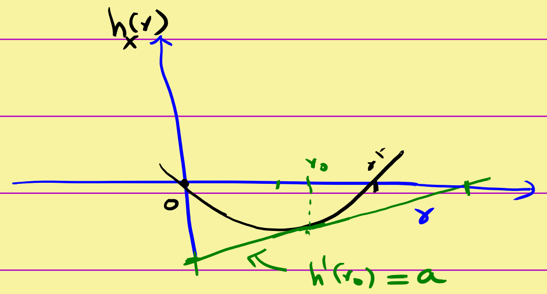
where $h_X(\gamma) = \log E[e^{\gamma X}]$ $\quad = e^{-\gamma n a + n h_X(\gamma)}$

$$\mu_X(a) = \inf\{h_X(\gamma) - \gamma a \mid \gamma > 0\}$$

$$\mathbb{P}(S_n \geq na) \leq e^{n \mu_X(a)}$$

Example: $X = \begin{cases} 1 & \text{wp } \frac{1}{2} \\ -1 & \text{wp } \frac{1}{2} \end{cases}$ $\quad E e^{\gamma X} = \frac{e^{\gamma} + e^{-\gamma}}{2}$

$$h_X(\gamma) = \log\left(\frac{e^{\gamma} + e^{-\gamma}}{2}\right)$$



Understand $\mu_X(a)$:- minimum of $h_X(\cdot) - \cdot a$

occurs at γ_0 :- $h_X'(\gamma_0) - a = 0$

$\boxed{\text{The picture is true in general}}$

where γ_0: $h_X'(\gamma_0) = a$

$h_X(\gamma) - \gamma a \big|_{\gamma=0} = 0$; $\quad \frac{\partial}{\partial \gamma}(h_X(\gamma) - \gamma a)\Big|_{\gamma=0} \text{"="} \frac{1}{E[e^{\gamma X}]} E[X e^{\gamma X}] - a \Big|_{\gamma=0} = E[X] - a$

< 0

if $E[X] < a$

- $\mathbb{P}(S_n \geq na) \leq e^{n\mu_X(a)} = e^{n[h_X(\gamma_0) - \gamma_0 h_X'(\gamma_0)]}$ (relevant if $a > E[X]$) as in this case $\mu_X(a) < 0$

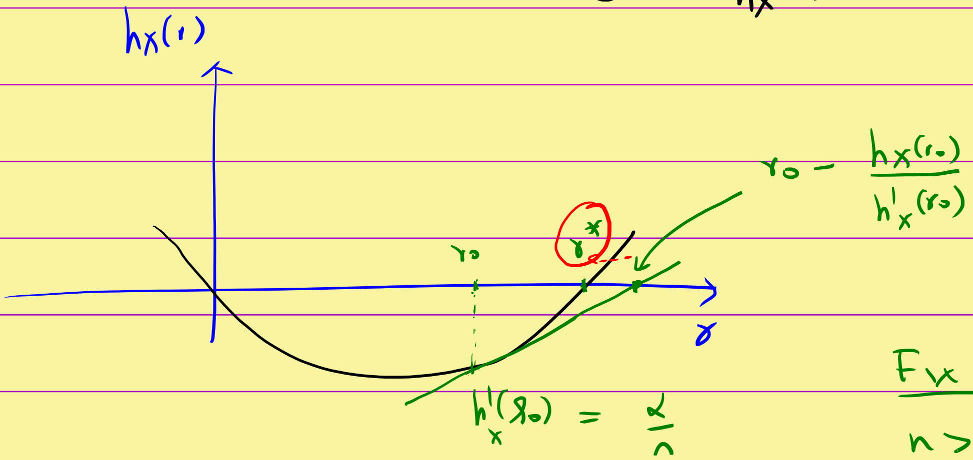
- Threshold Probabilities :- $\alpha > 0$, $E[X] < 0$, $E[e^{\gamma X}] < \infty \ \forall \gamma \in \mathbb{R}$

$$\mathbb{P}(S_n \geq \alpha)$$

$$\mathbb{P}\left(S_n \geq n\frac{\alpha}{n}\right) \leq e^{n[h_X(r_0) - r_0 h_X'(r_0)]}$$

where $h_X'(r_0) = \frac{\alpha}{n}$

$$= e^{\alpha\left[\frac{h_X(r_0)}{h_X'(r_0)} - r_0\right]}$$



Fix α

$n >>>$ $\qquad \frac{\alpha}{n} \equiv$ small

$n \to \infty$

$\boxed{n \to \infty}$

[J] $\qquad \mathbb{P}(S_n \geq \alpha) \leq e^{-\gamma^* \alpha}$

$$\boxed{\begin{array}{c} \gamma^* > 0 \\ h_X(\gamma^*) = 0 \end{array}}$$

$$\boxed{\frac{h_X(r_0)}{h_X'(r_0)} - r_0 \to -\gamma^*}$$

$$\boxed{EX < 0 \quad \& \quad \mathbb{P}(X > 0) > 0}$$

[A] $\qquad \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(S_n \geq na) = -I(a)$

$$I(a) = \sup\{at - h_X(t) \mid t \in \mathbb{R}\}$$

[J] $\qquad J = \inf\{n \geq 1 \mid S_n \geq \alpha, \ S_n \leq \beta\}$

$\beta < 0 < \alpha$

$$\mathbb{P}(S_J \geq \alpha) \leq e^{-\gamma^* \alpha}$$

Recall: Galton-Watson Process – Homework 9/10

[~1873] Sir Francis Galton [Cousin of Darwin]

Q: – survival of the English peerage [Educational times]
[male offsprings kept family name]

[Markov -(1906)]

Reverend Watson :- Solved Q in the same Journal [method]
– classic in text book.

· Ehrenfest [Paul / Tatiana] – thermodynamic reversibility [~1907]

– Modern foundations: Kolmogorov, Doeblin, Fréchet.

Galton-Watson Process :-

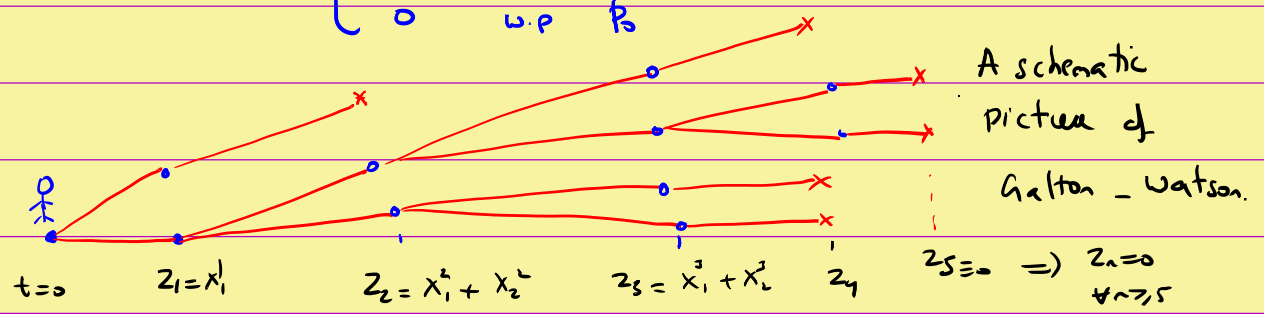
$$Z_n = \begin{cases} X_1^n + \cdots + X_{Z_{n-1}}^n & \text{if } Z_{n-1} > 0 \\ 0 & \text{if } Z_{n-1} = 0 \end{cases}$$

· $Z_0 = 1$ $\quad \forall n \geq 1$

· $\{X_i^n\}_{\{i \geq 1, n \geq 1\}}$ are i.i.d X

with $m = E[X]$ $\quad \sigma^2 = \mathrm{var}(X)$.

Example: $X = \begin{cases} 2 & \text{w.p } p_2 \\ 0 & \text{w.p } p_0 \end{cases}$



A schematic picture of Galton – Watson.

$$m = E[X] = 2p_2 = \begin{cases} 1 & p_2 = 1/2, \; p_0 = \frac{1}{2} \\ < 1 & p_2 < \frac{1}{2} \\ > 1 & p_2 > \frac{1}{2} \end{cases}$$

HW9: [Sketch of Solution] Re order series ✓

$n \geq 1$ · $E[Z_n] = \sum_{k \geq 1} E[Z_n \mathbb{1}(Z_{n-1} = k)]$

$$= \sum_{k \geq 1} E(Z_n \mid Z_{n-1} = k) \, \mathbb{P}(Z_{n-1} = k)$$

$$= \sum_{k \geq 1} E\Big[\sum_{i=1}^{Z_{n-1}} X_i^n \,\Big|\, Z_{n-1} = k\Big] \, \mathbb{P}(Z_{n-1} = k)$$

$$= \sum_{k \geq 1} E\Big[\sum_{i=1}^{k} X_i^n \,\Big|\, Z_{n-1} = k\Big] \, \mathbb{P}(Z_{n-1} = k)$$

independence & conditional expectation $\leftarrow$ $= \sum_{k \geq 1} m \, k \, \mathbb{P}(Z_{n-1} = k) = m \, E[Z_{n-1}]$

Induction $\Rightarrow$ $E[Z_n] = m^n$ $\quad \forall n \geq 1$

· $n > 1$ $\quad E[Z_n \mid \mathcal{F}_{n-1}] = m Z_{n-1}$ where $\mathcal{F}_{n-1}$

– observable events upto time $n-1$ by $\{Z_k\}_{k \geq 1}$

$\Rightarrow \left\{ \frac{Z_n}{m^n} \right\}_{n \geq 1}$ – was a martingale (non-negative)

• $\underline{m < 1}$

$$\mathbb{P}(Z_n > 0) \leq E[Z_n] = m^n$$

$$\Rightarrow \sum_{n=1}^{\infty} \mathbb{P}(Z_n > 0) < \infty$$

[Borel – Cantelli lemma] $\Rightarrow$ $\mathbb{P}(Z_n > 0 \text{ i.o.}) = 0$

If $Z_n = 0$ then $Z_m = 0 \ \forall m \geq n$ $\quad$ $\mathbb{P}(Z_n = 0 \text{ eventually}) = 1$

$\mathbb{P}(\exists m \geq 1 \ Z_m = 0) = 1$

Example

$$X = \begin{bmatrix} 2 & \text{w.p } 1/4 \\ 0 & \text{w.p } 3/4 \end{bmatrix}$$

$\mathbb{P}(\text{Galton–Watson process goes extinct}) = 1$

• $\underline{m = 1}$ $\quad \xi_n = \dfrac{Z_n}{m^n}$ is a martingale $\Rightarrow$ $\{Z_n\}_{n \geq 1}$ is a non-neg. martingale

By Corollary 3.17 : $\exists Z$ – r.v such that

$$\mathbb{P}(Z_n \to Z \text{ as } n \to \infty) = 1.$$

Ex:- $Z_n \in \mathbb{N} \cup \{0\}$ if $Z_n \to Z$ then $Z \in \mathbb{N}$

$$\mathbb{P}(\exists \underset{N(\omega)}{N} :- Z_n = Z \ \forall n \geq N) = 1. \quad — (*)$$

$\sigma = 0$ $\quad$ $\boxed{\cdot}$ $X = 1$ w.p. 1 $\quad$ [Deterministic : with $m = 1$]

$Z_0 \quad Z_1 \quad Z_2 \quad \cdots$ $\quad \Rightarrow Z = 1$

$p_0 > 0$

$\sigma \neq 0$ $\quad$ $\boxed{\cdot \ X \equiv \text{random}}$ $\quad \left[X = \{ k \text{ wp } p_k \text{ and } \sum_{k=0}^{\infty} k p_k = 1 \right]$

Only way $\{Z_n\}_{n \geq 1}$ "becomes constant" is if $Z_m = 0$ $\Rightarrow$ $Z_n = 0 \ \forall n \geq m$

$\vdots$ 3 particle $\quad$ $\boxed{\exists}$ 3 with positive probability

$k \quad k+1 \quad\quad N \quad N+1 \quad \cdots$

$$\Rightarrow \quad Z = 0$$

$\therefore \ \mathbb{P}(\exists N \cdot Z_n = 0 \ \forall n \geq N) = 1$

$\mathbb{P}(\text{Galton–Watson process goes extinct}) = 1$

Example :-

$$X = \begin{bmatrix} 2 & \text{wp } \tfrac{1}{2} \\ 0 & \text{wp } \tfrac{1}{2} \end{bmatrix}$$

$\underline{m>1}$:

$\underline{m = E[X]}$

Example :- $X = \begin{cases} 2 & \text{wp } 3/4 \\ 0 & \text{wp } 1/4 \end{cases}$

In general $\quad p_k \geq 1$; $p_0 > 0$

$X = k \quad \text{wp} \quad p_k \quad k \in \mathbb{N} \cup \{0\}$

$\sum_{k \geq 1} p_k = 1$

$\varphi : [0,1) \to [0,1]$ be the probability generating function

$$\phi(s) = \sum_{k \geq 0} p_k s^k$$

Ex: $\quad \phi'(s) = \sum_{k \geq 1} k \, p_k \, s^{k-1} \geq 0$

Ex:- $\quad \phi''(s) = \sum_{k \geq 2} k(k-1) \, p_k \, s^{k-2} \geq 0$

- $\phi(0) = p_0 \geq 0$, $\phi(1) = \sum_{k \geq 0} p_k = 1$
- $\phi'(1) = m > 1$ - assumption.

$\phi'(1) > 1$

(ϕ - convex)

$\phi'(1) > 1 \Rightarrow \exists\, \varepsilon_0 > 0. \quad \phi(1-\varepsilon) < 1-\varepsilon \quad \forall\, \varepsilon < \varepsilon_0$

$\underline{m>1} \Rightarrow \exists\, k_0 > 1 : p_{k_0} > 0 \Rightarrow \phi''(s) > 0 \quad \forall\, s > 0$

$\Rightarrow \phi$ is convex

Ex:- $\Rightarrow \exists\, \rho < 1 \quad \phi(\rho) = \rho$ and $\phi(s) < s \quad \forall\, s \in (\rho, 1)$

& $\exists!\, \rho \quad \phi(\rho) = \rho$

Ex: Suppose $\quad \phi_n(s) = \sum_{k=0}^{\infty} P(Z_n = k)\, s^k \qquad s \in [0,1)$

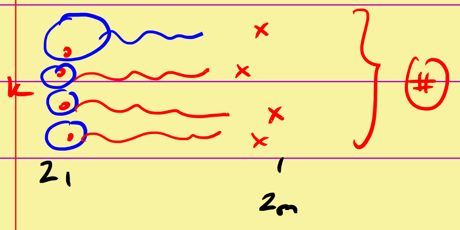
$\boxed{Z_n = \sum_{i=1}^{Z_{n-1}} X_i^n} \qquad \phi_n(s) = \underbrace{\phi \circ \cdots \circ}_{n\text{-times}} \phi(s)$

where $\phi(s) = \sum_{k \geq 0} p_k s^k$

$\alpha_n = P(Z_n = 0) = \sum_{k=0}^{\infty} P(Z_n = 0, Z_1 = k)$

$P(Z_1 = k | Z_0 = 1)$

$= \sum_{k=0}^{\infty} \underbrace{P(Z_n = 0 \mid Z_1 = k)}_{\#} \, p_k$



Ex $= \sum_{k=0}^{\infty} \left[P(Z_{n-1} = 0) \right]^k p_k$

$$\Rightarrow \boxed{\alpha_n = \sum_{k=0}^{\infty} \alpha_{n-1}^k \, p_k}$$

- i.e. $\alpha_n = \phi(\alpha_{n-1})$ (rewritten using p.g.f) — (##)

- $\alpha_n = \mathbb{P}(Z_n = 0) \leq \mathbb{P}(Z_{n+1} = 0) = \alpha_{n+1}$

α_n increasing sequence in $[0,1)$

let $\alpha_n \to \alpha$ as $n \to \infty$ for some $\alpha \in [0,1]$

by (##) $\Rightarrow$ [Ex: $\alpha_{n-1} \to \alpha$ as $n \to \infty$; $\phi(\cdot)$ — continuous]

$\alpha = \phi(\alpha)$

$\alpha = 1$ or $\alpha = \rho$

Now:

- $\alpha_0 = \mathbb{P}(Z_0 = 0) = 0$ [starting condition]
- $\phi(\rho) = \rho$ and $\alpha_m = \phi(\alpha_{n-1})$ ϕ is increasing

Induction :-

$\Rightarrow \alpha_n \leq \rho \quad \forall \; m \geq 1$

$\Rightarrow \alpha \leq \rho$

By above $\alpha = \rho$.

$$\alpha = \lim_{n \to \infty} \mathbb{P}(Z_n = 0) = \mathbb{P}(Z_k = 0 \text{ for some } k) = \rho < 1 \quad \rho \in (0,1)$$

lemma GW1: $m > 1$ $\mathbb{P}(Z_n > 0 \text{ for all } n) < 1$

let $\xi_n = \dfrac{Z_n}{m^n}$ $n \geq 1$ — martingale

Ex: $\sigma^2 < \infty \Rightarrow E[\xi_n^2] < \infty$

$$E\left((\xi_n - \xi_{n-1})^2 \mid A_{n-1} \right) = E\left(\xi_n^2 + \xi_{n-1}^2 - 2\xi_n \xi_{n-1} \mid A_{n-1} \right)$$

$$= E[\xi_n^2 \mid A_{n-1}] + \xi_{n-1}^2 - 2\xi_{n-1}\xi_{n-1}$$

$$\Rightarrow E[\xi_n^2 \mid A_{n-1}] = \xi_{n-1}^2 + E\left((\xi_n - \xi_{n-1})^2 \mid A_{n-1} \right)$$

Take Expectation and use Tower Property

$$\Rightarrow E[\xi_n^2] = E[\xi_{n-1}^2] + E[(\xi_n - \xi_{n-1})^2]$$

$$= E[\xi_{n-1}^2] + E\left[\left(\frac{Z_n}{m^n} - \frac{Z_{n-1}}{m^{n-1}}\right)^2\right]$$

$$= E[\xi_{n-1}^2] + m^{-2n} E[(Z_n - mZ_{n-1})^2]$$

$$\overset{Ex}{=} E[\xi_{n-1}^2] + m^{-2n} \sigma^2 E[Z_{n-1}]$$

$$\therefore E[\xi_n^2] = E[\xi_{n-1}^2] + \frac{\sigma^2}{m^{n+1}} \qquad (\because E[Z_n] = m^n)$$

$$\Rightarrow E[\xi_n^2] = 1 + \sigma^2 \sum_{k=1}^{n} \frac{1}{m^{k+1}}$$

$$\because m > 1 \Rightarrow \sup_{n \geq 1} E[\xi_n^2] < \infty \quad - \text{ (\#\#\#\#)}$$

$$\left(\sum_{k=1}^{n} \frac{1}{m^{k+1}} < \infty\right)$$

[Martingale Convergence Theorem] $\exists\ \xi$ r.v st

(####)

$$P(\xi_n \to \xi \text{ as } n \to \infty) = 1 \quad - (**)$$

$$\text{ie} \quad P\left(\frac{Z_n}{m^n} \to \xi \text{ as } n \to \infty\right) = 1$$

$\boxed{\text{Suppose } \xi \not\equiv 0}$ $\Longrightarrow$ " $\frac{Z_n}{m^n} \approx \xi$ as $n \to \infty$ "

$\Rightarrow$ $m > 1$ ($1 + Z_n > 0$) $\quad Z_n \to \infty$ as $n \to \infty$ with positive probability.

Fact: $P(Z_n \to \{0, \infty\} \text{ as } n \to \infty) = 1$

Extension of Martingale Convergence Theorem for Fact

$\xi_n = \frac{Z_n}{m^n}$ is a martingale.

let $\delta_n = \xi_n - \xi_{n-1}$

$$E[\xi_n^2] = E\left(\sum_{k=1}^{n} \delta_k + \xi_0\right)^2$$

$\langle \delta_n \rangle_{n \geq 1}$ are uncorrelated $\xleftarrow{Ex}$ $= \sum_{k=1}^{n} E\,\delta_k^2 + E[\xi_0^2]$ — (1)

We have shown $\sup_{n \geq 1} E\,\xi_n^2 < \infty$ (H)

$\therefore$ (1) $\Rightarrow$ $\sum_{k=1}^{\infty} E(d_k^2) < \infty$

As $E\left(\xi_{n+m} - \xi_n\right)^2 = \sum_{k=m+1}^{n+m} E(d_k^2)$

By (1)

$\therefore$ $\{\xi_n\}_{n \geq 1}$ is a Cauchy sequence in "L^2"

ie $\forall \epsilon > 0 \; \exists N \geq 1$ $E(\xi_n - \xi_m)^2 < \epsilon \quad \forall n, m > N$

along (xx) $\Rightarrow$ $E(\xi_n - \xi)^2 \longrightarrow 0$ as $n \to \infty$

(Ex) $\Longrightarrow$ $E\,\xi_n \longrightarrow E\,\xi$ as $n \to \infty$

$\therefore$ $E\,\xi_n = \frac{E\,Z_n}{m^n} = 1 \Rightarrow E[\xi] = 1$

$\Rightarrow$ $\xi \not\equiv 0$ r.v.

$\underline{m > 1}$

$$\mathbb{P}\left(\frac{Z_n}{m^n} \to \xi \text{ as } n \to \infty\right) = 1 \quad \text{— (xx)}$$

$$P(\xi > 0) > 0 \quad \text{— (xxx)}$$

$\Rightarrow$ $\mathbb{P}(Z_n \to \infty \text{ as } n \to \infty) = \mathbb{P}(\xi > 0)$

$\mathbb{P}(Z_n \to 0 \text{ as } n \to \infty) = \mathbb{P}(\xi = 0)$

$\Rightarrow$ $\mathbb{P}(Z_n \to \{\infty, 0\} \text{ as } n \to \infty) = 1$

$\square$

Cor: $p_0 = 0 \Rightarrow \mathbb{P}(Z_n \to \infty \text{ as } n \to \infty) = 1$

Discussion:

- $C[0,1] = \{ f : [0,1] \to \mathbb{R} \mid f \text{ - continuous} \}$

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

$(C([0,1]), d)$ - metric space

- $\mathcal{L} = \{ X \mid X : \Omega \to \mathbb{R} \quad (\Omega, \mathcal{F}, \mathbb{P}) \text{ - Probability space} \}$

X - random variable

Examples of d: $(\mathcal{L}, d)$ - metric space ?

X, Y

$$d(X,Y) = \| P_X - P_Y \|_{TV}$$

X, Y

$$d(X,Y) = \left(E|X-Y|^p \right)^{1/p}$$

$X_n \to X$ wp 1, $n \to \infty$

NOT metrizable

$X_n \xrightarrow{P} X$ as $n \to \infty$ ✓

$(\mathcal{L}, d_P)$

$d_P(X_n, X) \to 0$

$X_n \xrightarrow{d} X$ as $n \to \infty$ ✓

$(\mathcal{L}, d_F)$

$d_F(X_n, X) \to 0$ as $n \to \infty$